MATH 8803: Representation Theory II

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Contents

Lecture 1

Jan. 12 — Introduction and Review

1.1 Review and Overview

Remark. Recall that we are interested in representations of Lie groups G, which is closely related to representations of Lie algebras $\mathfrak{g}$.

We are primarily interested in semisimple Lie algebras. In this case, we fix a *Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$, where $r = \dim \mathfrak{h}$ is called the *rank*. We have the Serre generators $\{h_i, e_i, f_i\}_{i=1}^r$ and relations

$$[h_i, e_j] = a_{i,j} e_j, \quad [h_i, f_j] = a_{i,j} f_j, \quad \mathrm{ad}_{e_i}^{1-a_{i,j}} e_j = 0, \quad \mathrm{ad}_{f_i}^{1-a_{i,j}} f_j = 0,$$

where $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$ for $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$. Here $\{\alpha_i\} \subseteq \mathfrak{h}^*$ and we identify $\alpha_i^\vee \leftrightarrow h_i \in \mathfrak{h}$. Then

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where $\mathfrak{n}_+$ is generated by $\{e_i\}$ and $\mathfrak{n}_-$ is generated by $\{f_i\}$. We also have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R = R_+ \sqcup R_-$. We have $R_+ \subseteq Q_+$ and $R_- \subseteq Q_-$, where $Q_+ = \{\sum_{i=1}^r n_i \alpha_i : n_i \geq 0\}$. If the $a_{i,j}$ are degenerate, then we can define $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathbb{C}c$ is called the *central extension* and $d = t\frac{d}{dt}$. We can think of these as maps $S^1 \to \mathfrak{g}$.

We can also consider the universal enveloping algebra $U(\mathfrak{g})$, and the related object. $U_q(\mathfrak{g})$ We have an R-matrix $R_{V,W}$ for the representations $V \otimes W$ and $W \otimes V$, and we have the relation

$$R_{1,2} R_{1,3} R_{2,3} = R_{2,3} R_{1,3} R_{1,2}$$

in $V_1 \otimes V_2 \otimes V_3$. A main goal later in the course will be to relate the representations of $U_q(\mathfrak{g})$ and $\widehat{\mathfrak{g}}$.

In this case, we have the diagram:

$$\begin{tikzcd}
 & & \mathfrak{g} \arrow[dl] \arrow[dr] & & \\
 & U_q(\mathfrak{g}) \arrow[dl] \arrow[dr] & & \widehat{g} \arrow[ll] \arrow[dl] \arrow[dr] & \\
E_{q,\tau}(\mathfrak{g}) & & U_q(\widehat{g}) \arrow[ll] & & \widehat{\widehat{\mathfrak{g}}} \arrow[ll]
\end{tikzcd}$$

The object $U_q(\widehat{g})$ is related to quantum integrable models of spin chain type (XXX and XXZ), and $E_{q,\tau}(\mathfrak{g})$ is the *elliptic quantum group* (XYZ).

1.2 Representations of Semisimple Lie Algebras

Remark. Recall the *Weyl group* $W = \{s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha\}$. The *weight lattice* is

$$P = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \alpha \in R\} = \bigoplus_i \mathbb{Z}\omega_i,$$

where ω_i are the fundamental weights satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$.

We can consider the *highest weight representation.* The *Verma module* is $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the 1-dimensional representation of $U(\mathfrak{h} \oplus \mathfrak{n}_+)$ on which $\mathfrak{h}$ acts by $\lambda(h)$. Then

$$P(M_\lambda) = \lambda - \mathbb{Q}_+,$$

and for each $\lambda \in \mathfrak{h}^*$, M_λ has a unique irreducible quotient L_λ. The *dominant integral weights* λ satisfy

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+, \quad 1 \leq i \leq r,$$

where $\lambda = \sum_{i=1}^r n_i \omega_i$ with $n_i \in \mathbb{Z}_+$.

Theorem 1.1. *The finite-dimensional irreps of $\mathfrak{g}$ are classified up to isomorphism by $\lambda \in P_+$. Moreover, $P(V)$ is Weyl invariant, and for any $\mu \in P(V)$, $w \in W$,*

$$\dim L_\lambda[\mu] = \dim L_\lambda[w\mu].$$

Example 1.0.1. For $\mathfrak{g} = \mathfrak{sl}_2$, the dominant integral weights are $n \in \mathbb{Z}_{\geq 0}$, $L_n = V_n$, and the Weyl group W acts by reflection.

Remark (Weyl character formula)**.** Let $\chi_V(g) = \mathrm{tr}_V(g)$. We can represent $g \sim e^h$, where $h \in \mathfrak{h}$. Then

$$\chi_V(e^h) = \sum_{\mu \in P} (\dim V(\mu)) e^{\mu(h)}.$$

We can then formally define $\chi_V = \sum_{\mu \in P} (\dim V(\mu)) e^\mu$. The *Weyl character formula* is

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} (-1)^\ell (w) e^{w(\lambda+\rho)}}{\Delta},$$

where $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w) w\rho}$ is the *Weyl denominator*. Here $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum_{i=1}^r \omega_i$. The *Weyl dimension formula* is then

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

Recall the *Casimir operator* $\sum_{i=1}^{\dim \mathfrak{g}} x_i x^i \in U(\mathfrak{g})$, which acts by the scalar $(\lambda, \lambda + 2\rho)$.

1.3 Representations of SL_n and GL_n

Proposition 1.1. *For general simple $\mathfrak{g}$, let $\lambda = \sum_{i=1}^r m_i \omega_i$ be a dominant integral weight. Let $T_\lambda = \bigotimes_i L_{\omega_i}^{\otimes m_i}$ and $v = \bigotimes_i v_{\omega_i}^{\otimes m_i}$. Let V be the subrepresentation of T_λ generated by v. Then $V \cong L_\lambda$.*

Remark. For $\mathfrak{sl}_n$, we have $\lambda = \sum_{i=1}^{n-1} m_i\omega_i$. The Cartan subalgebra is

$$\mathfrak{h} = \mathbb{C}_0^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\}.$$

We have $\alpha_i^\vee = e_i - e_{i-1}$ and $\delta_{i,j} = (\omega_i, \alpha_j^\vee) = (\omega_i, e_j - e_{j+1})$, where $\omega_i = (1, \ldots, 1, 0, \ldots, 0)$ with i ones. We can associate λ with the partition

$$\lambda = (m_1 + \cdots + m_{n-1}, m_2 + \cdots + m_{n-1}, \ldots, m_{n-1}, 0) = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, 0),$$

and $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{n-1}$. Note that L_{ω_1} is the defining representation, where $v_{\omega_1} = (1, 0, \ldots, 0)^T = v_1$, where $\{v_1, \ldots, v_n\}$ is a basis of the defining representation. Then we have that $L_{\omega_m} = \wedge^m V$ with highest weight $v_1 \wedge \cdots \wedge v_m$. Here $e_i = E_{i,i+1}$. Then we see that $L_\lambda \subseteq \bigotimes_{i=1}^{n-1} (\wedge^i V)^{\otimes m_i}$.

Remark. To move to GL_n, we can write

$$\mathrm{GL}_n(\mathbb{C}) = (\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))/\mu_n,$$

where μ_n are the roots of unity embedded by $z \mapsto (z^{-1}, zI)$. We have a covering homomorphism

$$\begin{aligned} \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ (z, A) &\longmapsto zA. \end{aligned}$$

We need to determine the holomorphic representations of $\mathbb{C}^\times$. Its Lie algebra is spanned by h such that $e^{2\pi i h} = 1$. Within a representation, h acts by an operator H such that $e^{2\pi i H} = 1$. Thus all irreducible representations of $\mathbb{C}^\times$ are of the form $\chi_N(z) = z^N$. So for $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$, we have $L_{\lambda,N} = \chi_N \otimes L_\lambda$.

Exercise 1.1. Show that if $L_{\lambda,N} = \chi_N \otimes L_\lambda$, then $N = nr + \sum_{i=1}^{n-1} \lambda_i$ for some integer r.

Remark. Letting $m_n = r \geq 0$ in the above exercise, the representation $L_{\lambda, nm_n + \sum_{i=1}^{n-1} \lambda_i}$ for $\mathfrak{gl}_n$ corresponds to the partition $(m_1 + \cdots + m_n, \ldots m_{n-1} + m_n, m_n)$.

Remark. For SL_n, the representation $\wedge^n V$ is trivial, but it is the determinant for GL_n. For GL_n, we also have χ^k and $(\chi^*)^k = \chi^{-k}$, these are called the *polynomial representations*.

Remark. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \geq \cdots \geq \lambda_n$ be a partition with at most n parts. Then $|\lambda| = \sum_i \lambda_i$ is an eigenvalue of $1_n = \sum_{i=1}^n e_{i,i} \in \mathfrak{gl}_n$. We can realize λ as a Young diagram. Note that L_λ occurs in $V^{\otimes N}$, where V is the defining representation. We can decompose

$$V^{\otimes N} = \bigoplus_{\lambda : |\lambda| = N} L_\lambda \otimes \pi_\lambda,$$

where $\pi_\lambda = \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$. There is a natural action of S_N on $V^{\otimes N}$.

Theorem 1.2 (Schur-Weyl duality)**.** *Let A be the image of $U(\mathfrak{gl}_n)$ in $\mathrm{End}(V^{\otimes N})$ and B be the image of $\mathbb{C}S_N$ in $\mathrm{End}(V^{\otimes N})$. Then*

1. *the centralizer of A is B and vice versa;*
2. *if λ has at most n parts, then the representation π_λ of B (and hence of S_N) is irreducible, and such representations are pairwise non-isomorphic;*
3. *if $\dim V \geq N$, then the π_λ exhaust all irreducible representations of S_N.*

Lecture 2

Jan. 14 — Applications of Schur-Weyl Duality

2.1 The Schur Functor

Remark. Let V be the defining representation for GL_n. Then

$$V^{\otimes N} = \bigoplus_{\lambda : |\lambda| = N} L_\lambda \otimes \pi_\lambda.$$

Recall that if $\lambda = (\lambda_1, \dots, \lambda_n)$, then we have

$$\lambda_1 = m_1 + \cdots + m_n, \quad \lambda_2 = m_2 + \cdots + m_n, \quad \dots, \quad \lambda_n = m_n.$$

Definition 2.1. Suppose we are given the partition λ of N. The *Schur functor* S^λ is given by

$$S^\lambda V = \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N})$$

for a vector space V. Note that this language, we have $V^{\otimes N} = \bigoplus_\lambda S^\lambda V \otimes \pi_\lambda$.

Example 2.1.1. Consider the following:

1. $S^{(n)}V = S^n V$, where (n) is the partition of n with a single part.
2. $S^{(1^n)}V = \wedge^n V$, where (1^n) is the partition of n with n parts equal to 1.
3. $V \otimes V = S^{(2)}V \otimes \mathbb{C}_+ \oplus S^{(1,1)}V \otimes \mathbb{C}_-$, where $\mathbb{C}_2$ acts trivially on $\mathbb{C}_+$ and by the sign on $\mathbb{C}_-$.
4. $V \otimes V \otimes V = S^{(3)}V \otimes \mathbb{C}_+ \oplus S^{(2,1)}V \otimes \mathbb{C}^2 \oplus S^{(1,1,1)}V \otimes \mathbb{C}_-$, where S_3 acts trivially on $\mathbb{C}_+$ and by sign on $\mathbb{C}_-$ as before, and $\mathbb{C}^2 = \{(x, y, z) : x + y + z = 0\}$.

   Note that $V \otimes V = S^2 V \oplus \wedge^2 V$, so $S^2 V \otimes V = S^3 V \oplus S^{(2,1)}V$ and $\wedge^2 V \otimes V = \wedge^3 V \oplus S^{(2,1)}V$.

Remark. Let $\dim V = N$ and λ have k parts. Recall that by the Weyl dimension formula,

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

We have $R_+ = \{\alpha_{i,j} = e_i - e_j : i < j\}$ and $\rho = \sum_{i=1}^{N-1} \omega_i = (N-1, N-2, \dots, 1, 0)$ (recall that ω_i is i ones followed by zeros). Thus we see that

$$\dim S^\lambda V = \prod_{1 \le i < j \le N} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \le i < j \le k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \le i < k < j \le N} \frac{\lambda_i + j - i}{j - i}.$$

We can rewrite the second product as

$$\prod_{1\le<k<j\le N}\frac{\lambda_i+j-i}{j-i}=\prod_{i=1}^{k}\frac{(N+1-i)\cdots(N+\lambda_i-i)}{(k+1-i)\cdots(k+\lambda_i-i)}.$$

Proposition 2.1. *We have* $\dim S^\lambda V = P_\lambda(N)$*, where* P_λ *is a polynomial of degree* $|\lambda|$ *with rational coefficients and integer roots. The roots of* P_λ *are all integers from the interval* $[1-\lambda_1, k-1]$ *(occurring with multiplicities).*

Example 2.1.2. Let $P_n(N)$ correspond to S^nV. Then $\lambda_1 = n$ and $k = 1$, and

$$P_n(N)=\dim S^nV=\binom{N+n-1}{n}.$$

Similarly, one can see that

$$P_{1^n}(N)=\dim\wedge^nV=\binom{N}{n}.$$

One can also consider $P_{(a,b)}(N)$ corresponding to partitions with two parts. The values $P_{(a,n)}(N)$ are called the Narayana numbers, which are of use in combinatorics.

2.2 Invariant Theory

Remark. Let V be a finite-dimensional vector space and $\{T_i\}\in(V^*)^{\otimes m_i}\otimes V^{\otimes n_i}$ for $i=1,\dots,k$. One would like to characterize *invariants* of such collections, i.e. polynomial functions $F(T_1,\dots,T_k)$ which are invariant under the action of $\mathrm{GL}(V)$.

One can think of such a tensor in $(V^*)^{\otimes m_i}\otimes V^{\otimes n_i}$ as a vertex with m_i incoming edges and n_i outgoing edges. Then constructing invariants $\{T_i\}$ reduces to studying graphs where T_i corresponds to a vertex v_i of the graph Γ. This allows us to assign to a given graph an invariant function F_Γ.

Theorem 2.1. *The functions* F_Γ *for various* Γ *span the space of invariant functions.*

Proof. We can view an invariant as an invariant element of the space $\bigotimes_{i=1}^k((V^*)^{\otimes m_i}\otimes V^{\otimes n_i})$, which we can view as $\mathrm{End}_{\mathrm{GL}(n)}(V^{\otimes M},V^{\otimes N})$, where $M=\sum d_im_i$ (the number of incoming edges) and $N=\sum d_in_i$ (the number of outgoing edges). Note that this space is empty when $M\neq N$, and the statement follows by Schur-Weyl duality when $M=N$. □

Example 2.1.3. Let $m_i=n_i=1$. Then $T_1,\dots,T_k$ are matrices. Then the graph Γ must look like a cycle, hence the invariants are all of the form

$$F_{j_1,\dots,j_r}(T_1,\dots,T_k)=\mathrm{tr}(T_{j_1}\cdots T_{j_r}).$$

Note that these invariants are asymptotically algebraically independent (when V is large enough). In particular, if $P(T_1,\dots,T_k)=0$ in all dimensions, then $\mathrm{tr}(P(T_1,\dots,T_k)T_{k+1})=0$, which cannot be true as the trace decomposes in terms of the $F_{j_1,\dots,j_r}$. (However, note that $[X,Y]=0$ for 1×1 matrices and $[Z,[X,Y]^2]=0$ for 2×2 matrices.) This also implies the uniqueness of the μ_n in the BCH formula:

$$\log(\exp(x)\exp(y))=\sum_{n\ge1}\frac{\mu_n(x,y)}{n!}.$$

2.3 Weyl Character Formula for GL_n

Remark (Weyl character formula for GL_n)**.** Recall that Weyl's character formula gives

$$\chi_\lambda = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\sum_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2})}, \tag{$*$}$$

where the denominator is $\Delta = \prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2}) = \prod_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}$. Letting $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$,

$$\Delta = e^\rho \prod_{\alpha\in R_+}(1-e^{-\alpha}) = x_1^{n-1}x_2^{n-2}\cdots x_n^0\prod_{i<j}(1-x_j/x_i),$$

where $\rho = (n-1, n-2, \dots, 1, 0)$ and $x_i = e^{e_i}$ (e.g. $x_1 = e^{(1,0,\dots,0)}$). After multiplying we get that

$$\Delta = \prod_{i<j}(x_i - x_j).$$

On the other hand, using $\Delta = \prod_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}$, we have

$$\Delta = \sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)} = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{n-1}\cdots x_{s(n)}^0.$$

Comparing these two formulas, we recover the formula for the Vandermonde determinant:

$$\det(\{x_j^{n-i}\}_{1\le i,j\le n}) = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{n-1}\cdots x_{s(n)}^0 = \prod_{i<j}(x_i-x_j).$$

Now applying this to the numerator of $(*)$, we have

$$\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)} = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{\lambda_1+n-1}\cdots x_{s(n)}^{\lambda_n+0}.$$

Thus in total, the character χ_λ is given by

$$\chi_\lambda = \frac{\sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{\lambda_1+n-1}\cdots x_{s(n)}^{\lambda_n+0}}{\prod_{i<j}(x_i-x_j)} = \frac{\det(\{x_i^{\lambda_j+n-i}\})}{\prod_{i<j}(x_i-x_j)}.$$

These functions are known as the *Schur polynomials* $s_\lambda(x_1,\dots,x_n)$.

Example 2.1.4 (Character of S^mV)**.** Using the above formula, we get the identity

$$s_{(m)}(x_1,\dots,x_n) = \sum_{1\le j_1\le\cdots\le j_m\le n} x_{j_1}\cdots x_{j_m} = h_m(x_1,\dots,x_n),$$

the mth complete symmetric function.

Example 2.1.5 (Character of $\wedge^mV$)**.** Similarly, one gets the identity

$$s_{(1^m)}(x_1,\dots,x_n) = \sum_{1\le j_1<\cdots<j_m\le n} x_{j_1}\cdots x_{j_m} = e_m(x_1,\dots,x_n),$$

the mth elementary symmetric function.

Example 2.1.6 (Trace in $V^{\otimes N}$)**.** Consider $x \otimes \sigma$, where $x = \mathrm{diag}(x_1, \dots, x_n)$ and σ has m_i cycles of length i. Then we have

$$\mathrm{tr}_{V^{\otimes N}}(x \otimes \sigma) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

By Schur-Weyl duality, we have that

$$\mathrm{tr}_{V^{\otimes N}}(x \otimes \sigma) = \sum_\lambda \chi_\lambda(\sigma) s_\lambda(x) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Using the formula for the Schur polynomial, we get the identity

$$\sum_\lambda \chi_\lambda(\sigma) \det(\{x_i^{\lambda_j + N - j}\}) = \prod_{i<j} (x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Theorem 2.2 (Frobenius character formula)**.** *$\chi_\lambda(\sigma)$ is the coefficient of $x_1^{\lambda_1 + N - 1} \cdots x_N^{\lambda_N}$ in the polynomial*

$$\prod_{i<j} (x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

2.4 Howe Duality

Remark. Fix V, W and consider $S^n(V \otimes W)$, which is a representation of $\mathrm{GL}(V) \otimes \mathrm{GL}(W)$.

Theorem 2.3 (Howe duality)**.** *We have a decomposition*

$$S^n(V \otimes W) = \bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes S^\lambda W.$$

Proof. We can write

$$S^n(V \otimes W) = ((V \otimes W)^{\otimes n})^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n}.$$

Using Schur-Weyl duality for each part, we get that

$$\begin{aligned} S^n(V \otimes W) &= \left(\Big(\bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes \pi_\lambda \Big) \otimes \Big(\bigoplus_{\mu : |\mu| = n} S^\mu W \otimes \pi_\mu \Big) \right)^{S_n} \\ &= \bigoplus_{\lambda, \mu : |\lambda| = |\mu| = n} S^\lambda V \otimes S^\mu W \otimes (\pi_\lambda \otimes \pi_\mu)^{S_n}. \end{aligned}$$

Since $\pi_\lambda = \pi_\lambda^*$, by Schur's lemma we have $(\pi_\lambda \otimes \pi_\mu)^{S_n} = \mathbb{C}^{\delta_{\lambda,\mu}}$. □

Corollary 2.3.1 (Cauchy identity)**.** *Let $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_s)$. Then*

$$\sum_\lambda s_\lambda(x) s_\lambda(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - z x_i y_i}.$$

Lecture 3

Jan. 21 — Minuscule Weights

3.1 Minuscule Weights

Remark. Let $\mathfrak{g}$ be a simple complex Lie algebra.

Definition 3.1. A dominant integral weight ω for $\mathfrak{g}$ is called *minuscule* if $\langle \omega, \beta \rangle \leq 1$ for every positive coroot β (equivalently, if $|\langle \omega, \alpha \rangle| \leq 1$ for any coroot β).

Example 3.1.1. Clearly $\omega = 0$ is minuscule.

Example 3.1.2. Let $\mathfrak{g} = \mathfrak{sl}_n$ with fundamental weights $\{\omega_i\}_{i=1}^{n-1}$,[1] where

$$\omega_i = (\underbrace{1, \ldots, 1}_{i \text{ ones}}, 0, \ldots, 0)$$

Let $\alpha_{i,j} = \alpha_{i,j}^\vee = e_i - e_j$. Note that $\langle \omega_i, e_j - e_k \rangle = 0$ when $j, k \leq i$ or $j, k > i$, and $\langle \omega_i, e_j - e_k \rangle = 1$ when $j \leq i < k$. So all of the ω_i are minuscule in this case.

Lemma 3.1. *Every nonzero minuscule weight is fundamental.*

Proof. Suppose ω is minuscule. Then there exists i with $\langle \omega, \alpha_i^\vee \rangle = 1$. Moreover, there can only be one such i, since if there were many, then $\langle \omega, \theta^\vee \rangle \geq 2$, where $\theta^\vee$ is the longest coroot (i.e. if $\theta = \sum_{m_i > 0} m_i \alpha_i$ is the longest root, then $\theta^\vee = \sum_{m_i > 0} m_i \alpha_i^\vee$). So ω is necessarily fundamental. □

Example 3.1.3. For G_2, F_4, and F_8, none of the fundamental weights are minuscule.

Lemma 3.2. *A fundamental weight ω_i is minuscule if and only if $m_i = 1$ where $\theta^\vee = \sum_j m_j \alpha_j^\vee$.*

Proof. By the minuscule condition, we know $m_i \leq 1$. If $m_i = 1$, then for any positive coroot $\beta = \sum n_j \alpha_j^\vee$ we have $n_j \leq m_j$, so $n_i \leq 1$. Thus $\langle \omega_i, \beta \rangle = n_i \leq 1$, so ω_i is minuscule. □

Lemma 3.3. *If $\omega \in Q$ with $|\langle \omega, \beta \rangle| \leq 1$ for all coroots β, then $\omega = 0$.*

Proof. Assume to the contrary that $\omega = \sum_i \alpha_i \neq 0$. We may assume that $\sum_i |m_i|$ is smallest possible. Then $0 < (\omega, \omega) = \sum_i m_i (\omega, \alpha_i)$, since the form is positive definite. Thus there exists j such that m_j and $\langle \omega, \alpha_j^\vee \rangle$ have the same sign. By replacing ω with $-\omega$ if necessary, we may assume both are positive. Then $\langle \omega, \alpha_j^\vee \rangle = 1$. Consider the reflection $s_j(\omega) = \omega - \alpha_j = \sum_i m_i' \alpha_i$. So $m_i' = m_j - 1$ and $m_i' = m_i$. But then $\sum_i |m_i'| = \sum_i |m_i| - 1 < \sum_i |m_i|$, contradicting the minimality of ω. □

[1] Recall a *fundamental weight* is a weight ω_i such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for all simple coroots $\alpha_j^\vee$.

Proposition 3.1. *The following conditions are equivalent:*

1. *ω is minuscule;*
2. *all weights of L_ω belong to the Weyl orbit $W\omega$;*
3. *if λ is a dominant integral weight such that $\omega - \lambda \in Q_+$, then $\lambda = \omega$.*

Proof. ($1 \Rightarrow 3$) If $\omega = 0$, then $-\lambda \in Q_+$, so $(\lambda, \rho) \le 0$ where $\rho = \sum_{i=1}^r \omega_i$, so $\lambda = 0$. Now let $\omega = \omega_i$ be minuscule. Then $\omega_i - \lambda = \sum_k m_k \alpha_k$ with $m_k \ge 0$. If $m_k = 0$ for $k \ne i$, then the problem reduces to a lower rank Dynkin diagram. So we can assume $m_k > 0$ for every $k \ne i$. Let β be a positive coroot, then

$$\langle \omega_i - \lambda, \beta \rangle = \langle \omega_i, \beta \rangle - \langle \lambda, \beta \rangle \le \langle \omega_i, \beta \rangle \le 1.$$

If $\alpha_i^\vee$ does not occur in β, then the above is ≤ 0. In particular, we have $\langle \omega_i - \lambda, \alpha_j^\vee \rangle \le 0$ for $j \ne i$. If we also have $\langle \omega_i - \lambda, \alpha_i^\vee \rangle \le 0$, then $(\omega_i - \lambda, \omega_i - \lambda) \le 0$, so $\omega_i = \lambda$. Otherwise, $\langle \omega_i - \lambda, \alpha_i^\vee \rangle = 1$. Then $m_j > 0$ for every j, so $\langle \omega_i - \lambda, \theta^\vee \rangle \ge 1$, since $\theta^\vee$ is a dominant coweight. Then $\langle \lambda, \theta^\vee \rangle \le 0$, so we must have $\lambda = 0$ since $\theta^\vee$ contains all $\alpha_j^\vee$ with positive coefficients. But then $\omega_i \in Q$, which is impossible by Lemma 3.3.

($3 \Rightarrow 2$) If μ is any weight of L_ω, then there exists $w \in W$ such that $\lambda = w\mu$ is dominant (since every orbit of W intersects the dominant chamber at exactly 1 point). Then $\omega - \lambda \in Q_+$, so $\lambda = \omega$, hence $\mu = w^{-1}\omega \in W\omega$.

($2 \Rightarrow 1$) Suppose otherwise ω is not minuscule. Then $\langle \omega, \alpha^\vee \rangle > 1$ for some positive coroot $\alpha^\vee$. Then

$$2(\omega, \alpha) > (\alpha, \alpha).$$

Note that $\omega - \alpha$ is a weight of L_ω (weight of $f_\alpha v_\omega$, where v_ω is a highest weight vector and $\{e_\alpha, f_\alpha, \alpha^\vee\}$ is an $\mathfrak{sl}_2$-triple). But $\omega - \alpha$ is not W-conjugate to ω, since

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega)$$

but the pairing is W-invariant. Contradiction. □

Corollary 3.0.1. *If ω is minuscule, then $\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma$.*

3.2 Applications of Minuscule Weights

Proposition 3.2. *$\omega \in P_+$ is minuscule if and only if the restriction of L_ω to any root $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ is the direct sum of 1-dimensional and 2-dimensional representations.*

Proof. ($\Rightarrow$) Let ω be minuscule and $v \in L_\omega$ the highest weight vector (of weight $w\omega$) for $(\mathfrak{sl}_2)_\alpha$. Then

$$h_\alpha v = \langle w\omega, \alpha^\vee \rangle v = \langle \omega, w^{-1}\alpha^\vee \rangle v.$$

Then $h_\alpha v = 0$ or $h_\alpha v = v$, so the representation is 1-dimensional or 2-dimensional.

($\Leftarrow$) Suppose ω is not minuscule. Then there exists $\alpha \in Q_+$ with $\langle \omega, \alpha^\vee \rangle = m > 1$. Let v_ω be a highest weight vector, then $h_\alpha v_\omega = \langle \omega, \alpha^\vee \rangle v_\omega$, which leads to a higher-dimensional $\mathfrak{sl}_2$-representation. □

Corollary 3.0.2. *If ω is minuscule, then for every dominant integral weight λ of $\mathfrak{g}$, we have*

$$L_\omega \otimes L_\lambda = \bigoplus_{\gamma \in W\omega} L_{\lambda+\gamma}.$$

(It is assumed that if $\lambda+\gamma$ is not dominant, then $L_{\lambda+\gamma} = 0$.)

Proof. We know $\chi_\omega = \sum_{\mu \in W\omega} e^\mu$. Then we have

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\mu \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\rho)+\mu}}{\Delta} = \frac{\sum_{\gamma \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\gamma+\rho)}}{\Delta}$$

where Δ is the Weyl denominator. If $\lambda + \gamma \notin P_+$, then for some $\alpha_i^\vee$, we get $\langle \lambda+\gamma, \alpha_i^\vee \rangle < 0$. But we know $\langle \gamma, \alpha_i^\vee \rangle \geq -1$, so $\langle \lambda+\gamma, \alpha_i^\vee \rangle = -1$. Thus $\langle \lambda+\gamma+\rho, \alpha_i^\vee \rangle = 0$, so for any $w\gamma$, the term $ws_i\gamma$ comes with the opposite sign. So we get that

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega \,:\, \lambda+\gamma \in P_+} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\Delta} = \sum_{\gamma \in W\omega \,:\, \lambda+\gamma \in P_+} \chi_{\lambda+\gamma},$$

which proves the desired result. □

Example 3.1.4. For $\mathfrak{sl}_2$, we have $L_1 \otimes L_m = L_{m+1} \oplus L_{m-1}$, which leads to the formula

$$L_m \otimes L_n = \bigoplus_{k=|m-n|}^{m+n} L_k$$

Example 3.1.5. Let $V = V_{\omega_1}$ be the defining representation for GL_n. Then

$$L_{\omega_1} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + \square} L_\mu,$$

where λ is a partition and $\lambda + \square$ denotes the set of partitions obtained by adding a single box to λ. For example, for $\lambda = (3,3,2,1)$ we have

$$L_{\omega_1} \otimes S^{(3,3,2,1)}V = S^{(4,3,2,1)}V \oplus S^{(3,3,3,1)}V \oplus S^{(3,3,2,2)}V \oplus S^{(3,3,2,1,1)}V.$$

Similarly, for $\wedge^m V = L_{\omega_m}$ (where $\omega_m = (1,\dots,1,0,\dots,0)$ with m ones), we have

$$L_{\omega_m} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + m\square} L_\mu,$$

where we are allowed to add m boxes to λ in $\lambda + m\square$. For example,

$$\wedge^2 V \otimes S^{(3,1)}V = S^{(4,2)}V \oplus S^{(4,1,1)}V \oplus S^{(3,2,1)}V \oplus S^{(3,1,1,1)}V.$$

Lecture 4

Jan. 26 — Other Classical Lie Algebras

4.1 Applications of Minuscule Weights, Continued

Proposition 4.1. *We have the following:*

1. *Let λ be a partition of N. Then $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$.*
2. *Let μ be a partition of $N+1$. Then $\pi_\mu|_{S_N} = \bigoplus_{\lambda \in \mu - \square} \pi_\lambda$.*

Proof. (1) Let V be a vector space of sufficiently large dimension. By Frobenius reciprocity,

$$\mathrm{Hom}_{S_{N+1}}(\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda, V^{\otimes(N+1)}) \cong \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N} \otimes V) = V \otimes S^\lambda V.$$

Now by Schur-Weyl duality, we have

$$\mathrm{Hom}_{S_{N+1}}\Big(\bigoplus_{\mu \in \lambda + \square} \pi_\mu, V^{\otimes(N+1)}\Big) = \bigoplus_{\mu \in \lambda + \square} S^\mu V.$$

Since $V \otimes S^\lambda V = \bigoplus_{\mu \in \lambda + \square} S^\mu V$, we conclude that $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$.

(2) This is left as an exercise. Use a different version of Frobenius reciprocity. $\square$

Definition 4.1. Let λ be a partition, and $\lambda^\dagger$ be the *conjugate partition* (the one corresponding to the transposed diagram). For example, $(3,3,2,1)^\dagger = (4,3,2)$.

Corollary 4.0.1. *Let $\mathbb{C}_-$ be the sign representation of S_N. Then $\pi_\lambda \otimes \mathbb{C}_- \cong \pi_{\lambda^\dagger}$.*

Proof. This is left as an exercise. The proof is by induction on $N = |\lambda|$. Let $C = \sum_{i<j} (i\ j)$, and note that its eigenvalues are the same as the Casimir operator of SL_N. $\square$

Proposition 4.2 (Skew Howe duality)**.** *We have a decomposition $\wedge^n(V \otimes W) = \bigoplus_\lambda S^\lambda V \otimes S^{\lambda^\dagger} W$ (as $\mathrm{GL}(V) \otimes \mathrm{GL}(W)$-modules).*

Proposition 4.3. *Every coset in P/Q contains a unique minuscule weight. This gives a bijection between P/Q and minuscule weights, so the number of minuscule weights is equal to* $\det A$*, where A is the Cartan matrix.*

Proof. Let $C = a + Q \in P/Q$ be a coset. Let $\omega \in C \cap P_+$ be the element which minimizes $\langle \omega, \rho^\vee \rangle$. If λ is the dominant weight for L_ω, then $\lambda \in C \cap P_+$ implies that

$$(\lambda, \rho^\vee) \geq (\omega, \rho^\vee).$$

Thus $(\omega - \lambda, \rho^\vee) \leq 0$, so $\omega - \lambda \in Q_+$. Thus $\lambda = \omega$, so ω is minuscule. Now suppose $\omega_1, \omega_2 \in C$ are minuscule and $\omega_1 \neq \omega_2$ with $\omega_1 - \omega_2 \in Q$. By Lemma 3.3, we must have $\langle \omega_1 - \omega_2, \beta \rangle \geq 2$ for all coroots β. But then $\langle \omega_1, \beta \rangle = 1$ (which implies $\beta > 0$) and $\langle \omega_2, \beta \rangle = -1$ (which implies $\beta < 0$), a contradiction. $\square$

Remark. Let A be the Cartan matrix. For every root, we can write

$$\alpha_i = \sum_{j=1}^{r} A_{i,j} \omega_j.$$

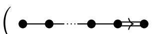
We have a covering map $\mathbb{R}^r/\Lambda_2 \to \mathbb{R}^r/\Lambda_1$, where $\Lambda_2 = P$ and $\Lambda_1 = Q$. Then $\det A$ is precisely the degree of this covering, which counts the number of cosets.

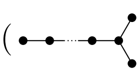
4.2 Other Classical Lie Algebras

Example 4.1.1. Recall that $\mathfrak{g} = \mathfrak{sp}_{2n}$ corresponds to the Dynkin diagram C_n (), where the arrow points from longer roots to shorter roots. We have $R_+ = e_i \pm e_j, 2e_j$. The simple roots are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \ldots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = 2e_n.$$

We have $\alpha_i^\vee = \alpha_i$ for $i \neq n$ and $\alpha_n^\vee = e_n$, and $\omega_i = (1, \ldots, 1, 0, \ldots, 0)$ (with i ones) for $1 \leq i \leq n$.

Example 4.1.2. The Dynkin diagram B_n () corresponds to $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Most things are the same as above, but we will have $\alpha_n = e_n$ and $\alpha_n^\vee = 2en$. We have the same ω_i for $i < n$, but we get $\omega_n = (1/2, \ldots, 1/2)$. We have $R_+ = e_i \pm e_j, e_i$.

Example 4.1.3. The Dynkin diagram D_n () corresponds to $\mathfrak{g} = \mathfrak{so}_{2n}$. In this case we have $R_+ = e_i \pm e_j$, and simple roots given by

$$\alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{n-2} = e_{n-1}, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_{n-1} + e_n.$$

We have $\omega_i = (1, \ldots, 1, 0, \ldots, 0)$ (with i ones) for $i = 1, \ldots, n-2$, but we get $\omega_{n-1} = (1/2, \ldots, 1/2, 1/2)$ and $\omega_n = (1/2, \ldots, 1/2, -1/2)$.

Remark. We have the following:

- For G_2, F_4, F_8, we have $\det A = 1$ (here A is the Cartan matrix), so the only minuscule weight is 0.
- For B_n, we have $\det A = 2$ (the nontrivial minuscule weight is $(1/2, \ldots, 1/2)$, and the representation has weights $(\pm 1/2, \ldots, \pm 1/2)$ with all possible combinations of $\pm$ and dimension 2^n).
- For D_n, we have $\det A = 4$. The minuscule weights are $\omega_1, \omega_{n-1}, \omega_n$. Here ω_1 is the $2n$-dimensional defining representation. The other two are spin representations of dimension 2^{n-1}, with weights $(\pm 1/2, \ldots, \pm 1/2)$, taking even or odd numbers of $-$ signs.

4.3 Representations of Symplectic Lie Algebras

Remark. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have the Dynkin diagram C_n and

$$\omega_i = (\underbrace{1, \ldots, 1}_{i \text{ ones}}, 0, \ldots, 0).$$

The elements of the Cartan subalgebra are given by $\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)$. So $L_{\omega_1} = V$ (the defining representation) with highest weight e_1. Note that $\wedge^2 V$ is not irreducible:

$$\wedge^2 V = \wedge_0^2 V \oplus \mathbb{C},$$

where $\mathbb{C}$ is the trivial representation spanned by $B^{-1} = \sum_i e_{i+n} \wedge e_i$ (note that B^{-1} is invariant under $\mathfrak{sp}_{2n}$. However, one can check that $\wedge_0^2 V$ is irreducible.

Now let us consider L_{ω_j} for $j \geq 2$. Let $B = \sum_i e_i^* \wedge e_{i+n}^*$. We have an operator

$$i_B : \wedge^{i+1} V \longrightarrow \wedge^{i-1} V,$$

and we can denote $\wedge_0^i V = \ker(i_B|_{\wedge^i V})$ (note that $i_B|_{\wedge^i V}$ is injective when $i \geq n$). The $\wedge_0^i V$ are irreducible for $i \leq n$, and one can check that these form all of the irreducible representations of $\mathfrak{sp}_{2n}$ (compute their dimensions and compare them to the highest weight representations).

We can also define an operator

$$\begin{aligned} m_B : \wedge^{i-1} V &\longrightarrow \wedge^{i+1} V \\ u &\mapsto B^{-1} \wedge u. \end{aligned}$$

One can check that m_B and i_B together with h (acting as $i - n$ on $\wedge^i V$) form an $\mathfrak{sl}_2$-triple. Then

$$\wedge V = \bigoplus_{i=0}^{n} L_{\omega_i} \otimes L_{n-j}$$

(where $\omega_0 = 0$ and L_{n-j} is the representation of $\mathfrak{sl}_2$ of weight $n - j$) as representations of $\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$.

4.4 Representations of Orthogonal Lie Algebras

Remark. First consider B_n, which corresponds to $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Let $Q = \sum_{i=1}^n x_i x_{i+n} + x_{2n+1}^2$. In this case, the Cartan subalgebra is given by elements of the form $\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n, 0)$. Let V be the $(2n+1)$-dimensional defining representation. Then for $1 \leq i \leq n-1$, the representation $\wedge^i V$ is irreducible (one can check this using the dimension formula) with highest weight

$$\omega_i = (\underbrace{1, \ldots, 1}_{i \text{ ones}}, 0, \ldots, 0).$$

On the other hand, $\wedge^n V$ is irreducible but not fundamental, with highest weight $(1, \ldots, 1) = 2\omega_n$.

Now we consider the spin representation S (whose elements are called *spinors*). It has weights

$$(\pm 1/2, \pm 1/2, \ldots, \pm 1/2)$$

(all possible combinations of $\pm$). The character of S is given by

$$\chi_S(x_1, \ldots, x_n) = (x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}).$$

Remark. We will want to look at the Lie group $\mathrm{Spin}_{2n+1}(\mathbb{C})$, the universal cover of $\mathrm{SO}_{2n+1}(\mathbb{C})$. For $n = 1$, we have $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$. We will see that S is 2-dimensional, and $\pi_1(\mathrm{SO}_3(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.

Lecture 5

Jan. 28 — Other Classical Lie Algebras, Part 2

5.1 More on Orthogonal Lie Algebras

Proposition 5.1. *For $n \geq 3$, we have $\pi_1(\mathrm{SO}_n(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. There is a deformation retract from the surface X_n defined by $z_1^2 + \cdots + z_n^2 = 1$ in $\mathbb{C}^n$ to the sphere $X_n^{\mathbb{R}} = X_n \cap \mathbb{R}^n$ defined by $x_1^2 + \cdots + x_n^2 = 1$ in $\mathbb{R}^n$: Let $\vec{z} = \vec{x} + i\vec{y} \in X_n$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$, and note that $|\vec{z}|^2 = 1$ if and only if $|\vec{x}|^2 - |\vec{y}|^2 = 1$ and $\vec{x} \cdot \vec{y} = 0$. We also have

$$(\vec{x} + ti\vec{y})^2 = |\vec{x}|^2 - t^2|\vec{y}|^2 = 1 + (1 - t^2)|\vec{y}|^2 \geq 1.$$

So we can define a homotopy $f_t : X_n \to X_n$ by

$$f_t(\vec{z}) = \frac{\vec{x} + ti\vec{y}}{\sqrt{|\vec{x}|^2 - t^2|\vec{y}|^2}},$$

which satisfies $|f_t(z)|^2 = 1$, $f_1(z) = z$, and $f_0(z) \in X_n^{\mathbb{R}}$. Now observe that SO_n acts on X_n with fibers isomorphic to SO_{n-1}, so we have a long exact sequence

$$\pi_2(X_n) \longrightarrow \pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \longrightarrow \pi_1(\mathrm{SO}_n(\mathbb{C})) \longrightarrow \pi_1(X_n).$$

The first and last groups are trivial for $n \geq 4$, so we have that $\pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \cong \pi_1(\mathrm{SO}_n(\mathbb{C}))$. Thus the result follows once one checks that $\pi_1(\mathrm{SO}_3(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$ (left as an exercise). $\square$

Remark. Now consider D_n, which corresponds to $\mathfrak{g} = \mathfrak{so}_{2n}$. Let $Q = \sum_{i=1}^n x_i x_{i+n}$. The elements of the Cartan subalgebra are given by $\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)$. Let V be the $2n$-dimensional defining representation, and consider $\wedge^i V$ for $1 \leq i \leq n$. We have $\wedge^i V$ is irreducible for $0 \leq i \leq n-1$, and $L_{\omega_i} = \wedge^i V$ for $1 \leq n-2$. Note that $L_{(1,\ldots,1,0)}$ is irreducible but not fundamental. Letting

$$\omega_{n-1} = (1/2, \ldots, 1/2, 1/2) \quad \text{and} \quad (1/2, \ldots, 1/2, -1/2),$$

the corresponding $S_+ = L_{\omega_{n-1}}$ and $S_- = L_{\omega_n}$ are the spin representations. The characters are

$$\chi_{S_\pm} = ((x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}))_\pm,$$

where the $\pm$ denotes an even or odd number of $-$ signs.

Example 5.0.1. We have $\mathrm{Spin}_4 = \mathrm{SL}_2 \times \mathrm{SL}_2$, where factors correspond to S_+ and S_-. We have $\mathrm{Spin}_5 = \mathrm{Sp}_4$, where S is the 4-dimensional defining representation, and $\mathrm{SO}_5 = \mathrm{Sp}_4/\{\pm 1\}$. We have $\mathrm{Spin}_6 = \mathrm{SL}_4$, where S_+, S_- are the 4-dimensional defining representation and its dual, and $\mathrm{SO}_6 = \mathrm{SL}_4/\{\pm 1\}$.

Example 5.0.2. Let V be a finite-dimensional vector space, and consider $SV = \mathbb{C}[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ is an orthonormal basis. Denote $R^2 = \sum_{i=1}^n x_i^2 = S^2V$ and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$. Then:

1. Find a first-order differential operator making $\{R^2, \Delta, \cdot\}$ an $\mathfrak{sl}_2$-triple. Make sure that it commutes with the $\mathrm{SO}(V)$ action.
2. Let $H_m \subseteq S^mV$ be the subspace of harmonic polynomials. Then

$$SV = \bigoplus_{m=0}^{\infty} H_m \otimes W_m,$$

where $H_m = L_{m\omega_1}$ is the irreducible representation of $\mathrm{SO}(V)$, and W_m is the Verma module for $\mathfrak{sl}_2$ of highest weight m.

5.2 Clifford Algebras

Definition 5.1. Let V be a finite-dimensional vector space (over $\mathbb{C}$) and $(\cdot, \cdot)$ a non-degenerate inner product on V. Give an associative algebra structure to V by

$$v^2 = \frac{1}{2}(v, v).$$

Such an algebra is called a *Clifford algebra*, and is denoted by $\mathrm{Cl}(V)$.

Corollary 5.0.1. $ab + ba = (a+b)^2 - a^2 - b^2 = (a, b)$.

Example 5.1.1. The operators ${}^i\partial/\partial x_i$ and $dx_i \wedge \cdot$ define a Clifford algebra.

Example 5.1.2. Let $e^ie^j + e^je^i = \delta_{i,j}$. Then $D = \sum_{i=1}^n e^i\partial_i$ (the *Dirac operator*) satisfies $D^2 = \Delta$.

Theorem 5.1. *The algebra* $\mathrm{Cl}(V)$ *is isomorphic to* $\mathrm{Mat}_{2^n}(\mathbb{C})$ *if* $\dim V = 2n$ *and to* $\mathrm{Mat}_{2^n}(\mathbb{C}) \oplus \mathrm{Mat}_{2^n}(\mathbb{C})$ *if* $\dim V = 2n+1$.

Proof. First consider the even case. Choose a basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that

$$(a_i, a_j) = (b_i, b_j) = 0, \quad (a_i, b_j) = \delta_{i,j}, \quad a_ia_j + a_ja_i = 0, \quad b_ib_j + b_jb_i = 0, \quad b_ia_i + a_ib_i = 1.$$

Consider $\mathrm{Cl}(V)$-module $M = \wedge(a_1, \ldots, a_n)$ (note that $\dim M = 2^n$) with action defined by

$$\rho(a_i)w = a_iw \quad \text{and} \quad \rho(b_i)w = \frac{\partial w}{\partial a_i}.$$

We have the relations

$$1 = \left[a_i, \frac{\partial}{\partial a_i}\right] = a_i\frac{\partial}{\partial a_i} + \frac{\partial}{\partial a_i}a_i \quad \text{and} \quad a_j\frac{\partial}{\partial a_i} = -\frac{\partial}{\partial a_i}a_j$$

for $i \neq j$. Let $c_{I,J} = a_{i_1} \cdots a_{i_k}b_{j_1} \cdots b_{j_m}$ for $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_m\}$. Check as an exercise that the $c_{I,J}$ are linearly independent, then $\rho : \mathrm{Cl}(V) \to \mathrm{End}(M)$ is an isomorphism.

If $\dim V = 2n+1$, then we can pick an extra element z satisfying

$$(z,a_i) = (z,b_i) = 0 \quad \text{and} \quad (z,z) = 2,$$

with relations $za_i + a_iz = zb_i + b_iz = 0$ and $z^2 = 1$. Then $zw = \pm(-1)^{\deg w} wz$ for $w \in M_\pm$. □

Remark. There is an embedding $\mathfrak{so}(V) \to \mathrm{Cl}(V)$. Define a map

$$\begin{aligned} \xi : \wedge^2 V = \mathfrak{so}(V) &\longrightarrow \mathrm{Cl}(V) \\ a \wedge b &\longmapsto \frac{1}{2}(ab - ba) = ab - \frac{1}{2}(a,b). \end{aligned}$$

One can check that $[\xi(a \wedge b), \xi(c \wedge d)] = \xi([a \wedge b, c \wedge d])$, so ξ is a homomorphism of Lie algebras. We have $\xi^* M$ for even dimensional V and $\xi^* M_\pm$ for odd dimensional V, and

$$\rho_{\xi^* M}(a) = \rho_M(\xi(a))$$

gives $\xi^* M$ the structure of an $\mathfrak{so}(V)$-representation (and similarly for $\xi^* M_\pm$. Notice that $\chi^* M$ is reducible:

$$\xi^* M = (\xi^* M)_0 \oplus (\xi^* M)_1$$

as representations, where the first factor corresponds to even degree and the second to odd degree.

Example 5.1.3. We have the following:

1. $(\xi^* M)_0 \cong S_+$ and $(\xi^* M)_1 \cong S_-$ for even dimensional V.
2. If $\dim V$ is odd, then $\chi^* M_\pm$ are both isomorphic to S.

Lecture 6

Feb. 2 — Duals, Maximal Weights, Exponents

6.1 Dual Representations

Remark. Let L_λ be the irreducible representation of highest weight λ. What is the highest weight of the dual representation L_λ^*? Let w_0 be the maximal element in W.

Proposition 6.1. *We have* $L_\lambda^* = L_{-w_0(\lambda)}$.

Proof. Since λ is the highest weight in L_λ, for every weight μ in L_λ we have $\lambda - \mu \in Q_+$. So

$$Q_- \ni w_0(\lambda - \mu) = w_0(\lambda) - w_0(\mu),$$

so $w_0(\mu) - w_0(\lambda) \in Q_+$. Thus $w_0(\lambda) \le w_0(\mu)$ for all $\mu \in L_\lambda$, so the length of w_0 is $|R_+|$. Thus $-w_0(\lambda)$ is the lowest weight of L_λ, which is the highest weight of L_λ^*. □

Example 6.0.1. Since the length of w_0 is $|R_+|$, w_0 permutes the fundamental (co)weights and (co)roots, so w_0 is an automorphism of Dynkin diagrams. Note that that W acts on P/Q, and w_0 acts as inversion.

- The Dynkin diagrams $A_1, B_n, C_n, G_2, F_4, E_7, E_8$ have no automorphisms, so $L_\lambda^* = L_\lambda$ for these.
- For A_n with $n \ge 2$, we have $P/Q = \mathbb{Z}/n\mathbb{Z}$ (e.g. if V is the defining representation, then we have that $L_{\omega_1}^* = V^* = \wedge^{n-1}V = L_{\omega_{n-1}}$).
- For E_6, we have $P/Q = \mathbb{Z}/3\mathbb{Z}$, where w_0 exchanges the two minuscule weights.
- For D_{2n+1}, we have $P/Q = \mathbb{Z}/4\mathbb{Z}$ and $S_+^* = S_-$. For D_{2n}, $P/Q = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $S_\pm^* = S_\pm$.

6.2 Maximal Weights

Definition 6.1. Let *maximal weight* of $\mathfrak{g}$, denoted θ, is the highest weight of the adjoint representation.

Example 6.1.1. If $\mathfrak{g} = \mathfrak{sl}_n$, then θ is the highest weight for $V^* \otimes V$ where V is the defining representation. Note that $V^* = \wedge^{n-1}V$, so the highest weight of $V^* \otimes V$ is $\theta = \omega_1 + \omega_{n-1}$. It is not fundamental.

Example 6.1.2. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $\mathfrak{g} = S^2V$ where V is the defining representation for $\mathfrak{sp}_{2n}$. Then $\theta = 2\omega_1$, which is also not fundamental.

Proposition 6.2. *For a simple Lie algebra with* $\mathfrak{g} \ne \mathfrak{sl}_n, \mathfrak{sp}_{2n}$*, the maximal weight* θ *is fundamental.*

Example 6.1.3. For $\mathfrak{so}_N$ with $N \ge 7$ (type B or D), we have $\mathfrak{g} = \wedge^2 V = L_{\omega_2}$.

6.3 Principal $\mathfrak{sl}_2$-Subalgebra and Exponents

Definition 6.2. Let $\mathfrak{g}$ be a simple Lie algebra and $\{e_i, f_i, h_i\}$ (where $h_i = \alpha_i^\vee$) be Chevalley generators. Let $e = \sum_{i=1}^r e_i$, and h such that $\alpha_i(h) = 2$ for all i (so $h = 2\rho^\vee$). Note that we have $[h, e] = 2e$ and $h = \sum_{i=1}^r (2\rho^\vee, \omega_i)\alpha_i^\vee$. Let $f = \sum_{i=1}^r (2\rho^\vee, \omega_i) f_i$. Then $\{h, e, f\}$ spans the *principal* $\mathfrak{sl}_2$*-subalgebra* of $\mathfrak{g}$.

Example 6.2.1. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Then the restriction of the defining representation to the principal $\mathfrak{sl}_2$ is L_n, the irreducible representation of $\mathfrak{sl}_2$ of highest weight n.

Remark. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, so that $\mathfrak{g} = \sum \mathfrak{g}[2m]$ where m is the height of the corresponding root subspace (and $2m$ is the weight with respect to h). Note $\mathfrak{g}[0] = \mathfrak{h}$ and $\dim \mathfrak{g}[0] = r$. Let $r_m = \dim \mathfrak{g}[2m]$.

Definition 6.3. We say that m is an *exponent* of $\mathfrak{g}$ if $r_m > r_{m+1}$. The *multiplicity* of an exponent m is $r_m - r_{m+1}$.

Remark. We have $r_0 = r$ and there are r exponents (counted with multiplicities) $m_1 \le m_2 \le \cdots \le m_r$. The roots of height 2 are given by $\alpha_i + \alpha_j$ (where i, j are connected in the in the Dynkin diagram). So $r_0 = r_1 = 1$ and $r_2 = r - 1$. Thus $m_1 = 1$ and $m_2 > 1$. We have

$$m_r = (\rho^\vee, \theta) = h_{\mathfrak{g}} - 1,$$

where θ is the highest root. We call $h_{\mathfrak{g}}$ the *Coxeter number* of $\mathfrak{g}$. Note that $\sum_{i=1}^r m_i = |R_+|$.

Proposition 6.3. *The restriction of* $\mathfrak{g}$ *to its principal* $\mathfrak{sl}_2$*-subalgebra decomposes as* $\bigoplus_{i=1}^r L_{2m_i+1}$.

Example 6.3.1. The exponents for $\mathfrak{sl}_n$ are $1, 2, \ldots, n-1$.

Definition 6.4. The *Coxeter number* of $\mathfrak{g}$ is $h_{\mathfrak{g}} = \langle \theta, \rho^\vee \rangle + 1 = m_r + 1$, and the *dual Coxeter number* is

$$h_{\mathfrak{g}}^\vee = \langle \widetilde{\theta}^\vee, \rho \rangle + 1,$$

where $\widetilde{\theta}^\vee = 2\theta/(\theta, \theta)$. If we normalize $(\theta, \theta) = 2$, then $h_{\mathfrak{g}}^\vee = \frac{1}{2}(\theta, \theta + 2\rho)$, which is the eigenvalue of $\frac{1}{2}C$ (where C is the Casimir operator).

6.4 Complex, Real, and Quaternionic Types

Definition 6.5. Let G be a Lie group. An irreducible representation V of G or $\mathfrak{g}$ is of *complex type* if $V \not\cong V^*$, *real type* if there exists a symmetric isomorphism $V \to V^*$ (i.e. a symmetric inner product for V), and *quaternionic (or symplectic) type* if the isomorphism is given through an anti-symmetric inner product.

Exercise 6.1. Let V be an irreducible representation of a finite group G. Show that $\mathrm{End}_{\mathbb{R}G}(V)$ (i.e. $V \otimes V^*$) can only be one of three types:

- complex type if $\mathrm{End}_{\mathbb{R}G}(V) \cong \mathbb{C}$,
- real type if $\mathrm{End}_{\mathbb{R}G}(V) \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$,
- quaternionic type if $\mathrm{End}_{\mathbb{R}G}(V) \cong \mathbb{H}$.

Example 6.5.1. Let L_n be an irreducible representation of $\mathfrak{sl}_2$. Then L_n is of real type for even n and of quaternionic type for odd n. Thus $L_n = S^n V$ where $V = L_1$ is 2-dimensional. The invariant form on $S^n V$ is $S^n B$, where B is a skew-symmetric invariant form on V.

Proposition 6.4. *Assume $\lambda = -w_0(\lambda)$, so that the corresponding representation is of real or quaternionic type. Then L_λ is of real type if $(2\rho^\vee, \lambda)$ is even and of quaternionic type if it is odd.*

Proof. The number $n = (2\rho^\vee, \lambda)$ is the eigenvalue of h (from the principal $\mathfrak{sl}_2$-subalgebra) on the highest weight vector. Thus we have a decomposition

$$L_\lambda|_{\mathfrak{sl}_2} = L_n \oplus \bigoplus_{m<n} k_m L_m,$$

where L_n has multiplicity 1. One can determine the type based on L_n. □

6.5 Review of Compact Lie Groups

Remark. Let G be a real Lie group of dimension n. Then $\xi \in \wedge^n \mathfrak{g}^*$ gives a generating n-form ω, which is non-vanishing if ξ is non-vanishing. This gives rise to left- and right-invariant measures μ_L and μ_R on G, which are unique up to a constant. We say that G is *unimodular* if $\mu_L = \mu_R$ (up to constants).

When does $\mu_L = \mu_R$? For a 1-dimensional representation V of G, let $|V|$ be the representation of G on the same space where $\rho_{|V|}(g) = |\rho_V(g)|$ (where $\rho_V : G \to \mathrm{Aut}(V) = \mathbb{R}^\times$).

Proposition 6.5. *We have $\mu_L = \mu_R$ if and only if $|\wedge^n \mathfrak{g}^*|$ is a trivial representation of G.*

Proof. We have $\mu_L = \mu_R$ if and only if the left-invariant form is right- or left-invariant up to a sign. This is equivalent to $\xi \in \wedge^n \mathfrak{g}^*$ being invariant up to a sign under the action of $\mathfrak{g}$. □

Proposition 6.6. *A compact group is unimodular.*

Proof. For compact groups, the representation $|\wedge^n \mathfrak{g}^*|$ gives a continuous homomorphism $G \to \mathbb{R}^+$, whose only compact subgroup is $\{1\}$. The result follows by Proposition 6.5. □

Proposition 6.7. *Let V be an irreducible representation of G. Then V admits a G-invariant unitary structure.*

Proof. Take any positive Hermitian form B on V, and define

$$B_{\mathrm{av}}(v, w) = \int_G B(\rho_V(g)v, \rho_V(g)w)\, dg.$$

This is well-defined and invariant by construction. □

Corollary 6.0.1 (Weyl unitary trick)**.** *Any finite-dimensional representation is completely reducible.*

Proof. Write $V = W \oplus W^\perp$. If W is invariant, then so is $W^\perp$. □

Lecture 7

Feb. 4 — Compact Groups

7.1 More on Exponents

Theorem 7.1 (Chevalley's restriction theorem)**.** *There is an isomorphism* $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}]^W$.

Theorem 7.2 (Harish-Chandra theorem)**.** *There is an isomorphism* $\mathbb{C}[\mathfrak{h}]^W \xrightarrow{\cong} \mathcal{Z}(U(\mathfrak{g}))$.

Remark. Pick an ordering $s_{i_1}, \ldots, s_{i_r}$ of the simple roots. Then $c = s_{i_1} \cdots s_{i_r}$ is the *Coxeter element*, and $c^h = 1$ where h is the Coxeter number. Then the eigenvalues of c are ζ^{m_i+1} where $\zeta = e^{2\pi i/h}$ and the m_i are the exponents. Also note that $|W| = \prod_{i=1}^r (m_i + 1)$.

If e, f, h is the principal $\mathfrak{sl}_2$-triple, then one can consider $e + \mathfrak{g}^f$ where $\mathfrak{g}^f = \ker \operatorname{ad}_f$.

7.2 Matrix Coefficients

Remark. For the rest of this lecture, let G be a real compact group and V a finite-dimensional continuous complex representation of G.

Definition 7.1. A *matrix coefficient* of $\rho_V : G \to \mathrm{GL}(V)$ is a function $G \to \mathbb{C}$ of the form

$$g \longmapsto \langle f, \rho_V(g)v \rangle$$

for some $v \in V$ and $f \in V^*$.

Proposition 7.1. *Matrix coefficients are smooth.*

Proof. Call $v \in V$ a smooth vector if $\langle f, \rho_V(g)v \rangle$ is smooth for all $f \in V^*$. It is obvious that such vectors form a subspace of V, call it $V_{\mathrm{sm}} \subseteq V$. Fix $v \in V$ and $\phi : G \to \mathbb{C}$ smooth and with compact support. Let

$$w = w(\phi, v) = \int_G \phi(g)\rho_V(g)v \, dg.$$

We claim that w is smooth. We have

$$f(\rho(h)w) = f\left(\rho_V(h) \int_G \phi(g)\rho_V(g)v \, dg\right) = \int_G f(\phi(g)\rho_V(hg)v) \, dg = \int_G f(\phi(h^{-1}g)\rho_V(g)v) \, dg.$$

Differentiating under the integral sign and noting that $\phi(h^{-1}g)$ is smooth in h, we see that the above expression is smooth in h. Now choose a delta-like sequence ϕ_n with compact support around 1 so that

$$\int_G \phi_n(g) \, dg = 1.$$

Then $w_n = w(\phi_n, v) \to v$ and each w_n is smooth, so v is smooth. $\square$

Remark. Let V be an irreducible representation of G. Then:

1. V has an invariant positive-definite inner product which is unique up to scaling;
2. one can use an orthonormal basis $v_1, \dots, v_n$ to define matrix coefficients:
$$\psi_{V,i,j}(g) = v_j^*(\rho_V(g)v_i) = (\rho_V(g)v_i, v_j)$$
(note that this definition is independent of normalization).

Theorem 7.3 (Orthonormality of matrix coefficients)**.** *Let V, W be irreducible representations of G.*

1. *If V, W are not isomorphic, then*
$$\int_G \psi_{V,i,j}(g)\overline{\psi}_{W,k,\ell}(g)\,dg = 0.$$
2. *For $V = W$, we have*
$$\int_G \psi_{V,i,j}(g)\overline{\psi}_{V,k,\ell}(g)\,dg = \frac{\delta_{i,k}\delta_{j,\ell}}{\dim V}.$$

Proof. Let $\{v_i\}$ and $\{w_k\}$ be orthonormal bases for V and W, respectively. We have
$$\int_G \psi_{V,i,j}(g)\overline{\psi}_{W,k,\ell}(g)\,dg = \int_G ((\rho_V(g) \otimes \rho_{\overline{W}}(g))(v_i \otimes w_k), v_j \otimes w_\ell)\,dg$$
Define the operator
$$P = \int_G (\rho_V \otimes \rho_{\overline{W}})(g)\,dg = \int_G \rho_{V \otimes \overline{W}}(g)\,dg.$$
Since $\overline{W} \cong W^*$, we have $P : V \otimes W^* \to V \otimes W^*$. Thus
$$\operatorname{Im} P \subseteq (V \otimes W^*)^G,$$
which is 0 if $V \ncong W$. On the other hand, if $V \cong W$, then the only invariant is
$$\vec{u} = \sum_k (v_k \otimes \overline{v}_k),$$
so P is the orthogonal projection onto $\vec{u}$. Thus
$$P\vec{x} = \frac{(\vec{x}, \vec{u})}{(\vec{u}, \vec{u})}\vec{u},$$
so we have $(P(v_i \otimes w_k), v_j \otimes w_\ell) = \delta_{i,j}\delta_{k,\ell}/(\dim V)$. $\square$

7.3 Peter-Weyl Theorem

Theorem 7.4 (Peter-Weyl theorem)**.** *The matrix coefficients $\psi_{V,i,j}$ form an orthogonal basis in $L^2(G)$.*

Remark. Let V be a finite-dimensional irrep of G. There is a natural inclusion
$$\begin{aligned} i_V : V^* &\hookrightarrow \operatorname{Hom}_G(V, L^2(G)), \\ f &\longmapsto [v \mapsto (\rho_{V^*}(\cdot)f)(v)]. \end{aligned}$$
We claim that i_V is also surjective. To see this, let $\phi \in \operatorname{Hom}_G(V, L^2(G))$, i.e. an L^2 function left-invariant

under G. Thus we have that

$$\phi(x) = \rho_{V^*}(xg^{-1})\phi(g)$$

(after modifying ϕ on a set of measure zero). Setting $g = 1$, we get $\phi(x) = \rho_{V^*}(x)\phi(1)$, so we have

$$\xi : \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes V^* \cong \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes \mathrm{Hom}_G(V, L^2(G)) \hookrightarrow L^2(G),$$

an embedding of $(G \times G)$-modules. Call the left-hand side $L^2_{\mathrm{alg}}(G)$.

Theorem 7.5 (Peter-Weyl theorem, alternative)**.** *$L^2_{\mathrm{alg}}(G)$ is dense in $L^2(G)$, i.e.*

$$L^2(G) = \widehat{\bigoplus}_{V \in \mathrm{Irr}(G)} V \otimes V^*.$$

Example 7.1.1. Let $G = S^1 = U(1)$. The irreducible representations of G are $\psi_n(\theta) = e^{in\theta}$. The $e^{in\theta}$ form a basis of $L^2(G) = L^2(S^1)$, where the norm is given by

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 \, d\theta.$$

This is the usual Fourier series on S^1. The Peter-Weyl theorem extends this to non-abelian groups.

Exercise 7.1. Let G be a compact group and H a closed subgroup of G.

1. Show that $L^2(G/H) = \widehat{\bigoplus}_{V \in \mathrm{Irr}(G)} N_H(V) V$, where $N_H(V) = \dim V^H$ (the space of H-invariants).

2. Let $G = \mathrm{SO}(3)$ and $H = \mathrm{SO}(2)$. Then show that $L^2(G/H) = L^2(S^2) = \widehat{\bigoplus}_{m \geq 0} N_H(m) L_{2m}$, and that $N_H(m) = 1$ for every m.

7.4 Introduction to Quantum Mechanics

Remark. Let $\mathcal{H}$ be a Hilbert space and H a self-adjoint operator on $\mathcal{H}$. The spectrum of H gives the *energy levels* of the system. The elements $\psi(x, y, z) \in L^2(\mathbb{R}^3)$ are called *wave functions*, and we assume that they are normalized so that $\|\psi\|_{L^2} = 1$. This is so that

$$|\psi(x, y, z)|^2 \Delta V$$

gives the probability of a quantum particle to be in the region ΔV.

In general, there is also a time dependence in the wave function ψ, so we have $\psi(x, y, z, t)$. The time dependence is governed by the Schroedinger equation:

$$i\partial_t \psi = H\psi.$$

One can solve this equation via separation of variables, and we can write

$$\psi(x, y, z, t) = \sum_N e^{-iE_N t} \psi_N(x, y, z),$$

where the ψ_N are eigenvectors satisfying $H\psi_N = E_N \psi_N$.

Example 7.1.2. For the hydrogen atom, we have

$$H = -\frac{1}{2}\Delta - \frac{1}{r},$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplacian and $r = \sqrt{x^2 + y^2 + z^2}$. The $\Delta/2$ is called the *kinetic part* of H, and the $1/r$ is called the *potential part* of H.

Lecture 8

Feb. 9 — Hydrogen Atom

8.1 Bound States of the Hydrogen Atom

Remark. We are looking for eigenvectors for $H = -\frac{1}{2}\Delta - \frac{1}{r}$, i.e. $\psi_N \in L^2(\mathbb{R}^3)$ such that $H\psi_N = E_N\psi_N$ with $E_N < 0$. We first write the Laplacian in spherical coordinates:

$$\begin{aligned}\Delta &= \Delta_r + \frac{1}{r}\Delta_{\text{sph}} \\ \Delta_r &= \partial_r^2 + \frac{2}{r}\partial_r \\ \Delta_{\text{sph}} &= \frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\,\cdot\,),\end{aligned}$$

where ϕ is the angle in the xy-plane and θ is the angle from the positive z-axis. Then we have

$$\partial_r^2\psi + \frac{2}{r}\partial_r\psi + \frac{2}{r}\psi + \frac{1}{r^2}\Delta_{\text{sph}}\psi = -2E\psi,$$

which is solved by $\psi(r, \vec{u}) = f(r)\xi(\vec{u})$ for $\vec{u} \in S^2$ satisfying

$$\Delta_{\text{sph}}\xi + \lambda\xi = 0 \qquad (*)$$

$$f''(r) + \frac{2}{r}f'(r) + \left(\frac{2}{r} - \frac{\lambda}{r^2} + 2E\right)f(r) = 0.$$

Note that $(*)$ implies Δ_{sph} is rotationally invariant. By the Peter-Weyl theorem, we have

$$L^2(S^2) = \widehat{\bigoplus}_{\ell\geq 0} L_{2\ell},$$

where $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ and $L_{2\ell}$ are the irreps of SO(3).

Let $Y_\ell^0 \subseteq L_{2\ell}$ be a vector of weight 0, which is invariant under SO(2). Thus it depends only on θ. So we can write $Y_\ell^0(\theta) = P_\ell(\cos\theta)$, where P is a polynomial of degree ℓ. By orthogonality,

$$\int_{-1}^{1} P_k(z)P_\ell(z)\,dz = 0, \quad k \neq n.$$

Thus we can write

$$-\lambda_e P_\ell(z) = \Delta_{\text{sph}}P_\ell(z) = \partial_z(1 - z^2)\partial_z P_\ell(z).$$

From looking at the leading term we must have $\lambda_\ell = \ell(\ell + 1)$.

Now take $Y_\ell^m \in L_{2\ell}$ for $-\ell \le m \le \ell$. Write $Y_\ell^m(\phi,\theta) = e^{im\phi}P_\ell^m(\cos\theta)$. So we have

$$\frac{-m^2}{1-z^2}P_\ell^m + \partial_z(1-z^2)\partial_z P_\ell^m + \ell(\ell+1)P_\ell^m = 0, \quad -\ell \le m \le \ell.$$

This equation has a unique solution (up to scaling) on $[-1,1]$, given by

$$P_\ell^m = (1-z^2)^{m/2}\partial_z^{\ell+m}(1-z^2)^\ell.$$

Now we return to the radial equation:

$$f''(r) + \frac{2}{r}f'(r) + \left(\frac{2}{r} - \frac{\ell(\ell+1)}{r^2} + 2E\right)f(r) = 0.$$

Write $f(r) = r^\ell e^{-r/n}h(2r/n)$, where n is to be chosen later and h satisfies

$$\rho h''(\rho) + (2\ell+2-\rho)h'(\rho) + \left(n - \ell - 1 + \frac{1}{4}(1+2En^2)\rho\right)h(\rho) = 0.$$

Now choose $n = 1/\sqrt{-2E}$, so that $E = -1/2n^2$. Then the above equation becomes

$$\rho h''(\rho) + (2\ell+2-\rho)h'(\rho) + (n-\ell-1)h(\rho) = 0.$$

This equation is known as the *generalized Laguerre equation*. To get $\|\psi\|_{L^2}^2 < \infty$, we must have

$$\int_0^\infty \rho^{2\ell+2}e^{-\rho}|h(\rho)|^2\,d\rho < \infty,$$

where the extra $+2$ in $\rho^{2\ell+2}$ comes from the Jacobian. Solutions around 0 behave like $\rho^s(1+o(1))$, so

$$s(s+2\ell+1) = 0.$$

Thus either $s = 0$ or $s = -2\ell-1$.

First consider when $\ell = 0$. Then $s = -1$ and we have $\rho^{-1}(1+o(1))$, so $\psi \sim 1/r$ as $r \to 0$. Then

$$H\psi = E\psi + C\delta_0,$$

where δ_0 is the delta function at 0, so we do not get an eigenvector in this case.

Thus $s = -2\ell-1$. Expanding $h(\rho)$ in a series and substituting, we get the recursive formula

$$h_n(\rho) = \sum_{k=0}^\infty \frac{(1+\ell-n)\cdots(k+\ell-n)}{(2\ell+2)\cdots(2\ell+1+k)k!}\rho^k.$$

This series converges, and we have

$$\lim_{\rho\to\infty}\frac{h_n(\rho)}{\rho} = 1$$

unless the series terminates. Thus $n-\ell-1 \in \mathbb{Z}_{\ge 0}$, so we can write

$$h_n(\rho) = \sum_{k=0}^{n-\ell-1} \frac{(1+\ell-n)\cdots(k+\ell-n)}{(2\ell+2)\cdots(2\ell+1+k)k!}\rho^k = L_{n-\ell-1}^{2\ell+1}(\rho),$$

which is known as the *generalized Laguerre polynomial*:

$$L_N^\alpha(\rho) = \sum_{k=0}^N (-1)^N \frac{N\cdots(N-k+1)}{(\alpha+1)\cdots(\alpha+k)}\frac{\rho^k}{k!}.$$

Theorem 8.1. *The bound states (i.e. solutions to $H\psi = E\psi$ in $L^2(\mathbb{R}^3)$) of the hydrogen atom are*

$$\psi_{n,\ell,m}(r,\phi,\theta) = r^\ell e^{-r/n} L^{2\ell+1}_{n-\ell-1}(2r/n) Y^m_\ell(\theta,\phi),$$

where $n \in \mathbb{Z}_{>0}$, ℓ is an integer from $0,\dots,n-1$, $E_n = -1/2n^2$, and m is an integer between $-\ell,\dots,\ell$.

Remark. In Theorem 8.1, n is known as the *principal quantum number*, and ℓ is known as the *azimuthal quantum number*. Note that if $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = -\Delta_{\text{sph}}$, where iL_x, iL_y, iL_z are the generators of $\mathfrak{so}(3)$ satisfying $[L_{\{x}, L_y] = -iL_{z\}}$, then $\vec{L}^2 = C = \ell(\ell+1)$ is the Casimir operator.

Corollary 8.1.1. *The space W_n of states with principal number n has dimension n^2.*

Proof. This follows from $\sum_{\ell=0}^{n-1}(2\ell+1) = n^2$. □

Remark. Note that $\widehat{\bigoplus}_n W_n$ forms a proper, closed subspace $L^2_0(\mathbb{R}^3)$ of $L^2(\mathbb{R}^3)$. We need to find all φ with $(H\varphi,\varphi) \geq 0$ to reconstruct all of $L^2(\mathbb{R}^3)$. This corresponds to the continuous spectrum of H.

8.2 Spin

Remark. *Spin* is a kind of intrinsic angular momentum. Instead of just $L^2(\mathbb{R}^3)$, we should consider

$$L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 = L^2(\mathbb{R}^3) \otimes L_1$$

to be the space of states for the hydrogen atom. We have

$$V_n = (L_0 \oplus L_2 \oplus \cdots \oplus L_{2n-2}) \otimes L_1 = 2L_1 \oplus 2L_3 \oplus \cdots \oplus 2L_{2n-3} \oplus 2L_{2n-1},$$

so $\dim V_n = 2n^2$. We have an additional *spin operator* given by

$$S_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},$$

which acts on $\mathbb{C}^2$ in the standard basis e_+, e_-. Then we have

$$\psi_{n,\ell,m,+} = \psi_{n,\ell,m} \otimes e_+ \quad \text{and} \quad \psi_{n,\ell,m,-} = \psi_{n,\ell,m} \otimes e_-.$$

The *total spin* is $m + s$ (where s is the eigenvalue for S_z), which is either $m + 1/2$, or $m - 1/2$.

8.3 Pauli Exclusion Principle

Remark. The space $\wedge^k V_n$ corresponds to the space of states for k electrons at energy level n. Note that we must have $k \leq 2n^2$ to have $\wedge^k V_n \neq 0$, which gives the *Pauli exclusion principle.*

In the periodic table, one has *orbitals* s, p, d, f corresponding to $\ell = 0, 1, 2, 3$, respectively, written with coefficient n and with exponent k corresponding to the number of electrons in the orbital. For example, the element Ruthenium has

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^7 5s^1.$$

The periodic table is organized as follows: from left to right ordered by how many *valent* electrons (i.e. the number of electrons in the outermost orbital), and from top to bottom ordered by how many energy levels. For Ruthenium, it is on column 8 and row 5. The number is 44, for 44 total electrons.

Exercise 8.1. Let $\vec{r} = (x, y, z)$ and $\vec{p} = (-\partial_x, -i\partial_y, -i\partial_z)$ be the *position* and *momentum* operators. Let $\vec{L} = \vec{r} \times \vec{p}$ and $H = \frac{1}{2}\vec{p}^2 + U(r)$, where U is rotationally invariant. Show that:

1. The components $i\vec{L}$ are generators of the rotations on $\mathbb{R}^3$, and $[\vec{L}, \vec{p}^2] = 0$.
2. $\vec{A}_0 = \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p})$ satisfies $[\vec{A}_0, \vec{p}^2] = 0$.
3. Let $A = \vec{A}_0 + \phi(r)\vec{r}$. There exists ϕ such that $[\vec{A}, H] = 0$ if and only if U is a *Columb potential* (i.e. $U(r) = \frac{C}{r} + D$), and in this case ϕ is completely determined.
4. (Hidden symmetry of the hydrogen atom) Use the commutation relations between $\vec{A}$ and $\vec{L}$ to define an action on $\mathfrak{so}_4 = \mathfrak{so}_3 \oplus \mathfrak{so}_3$, so that $\vec{L}$ is the diagonal copy in this decomposition.
5. $W_n = L_{n-1} \boxtimes L_{n-1}$ as representations of $\mathfrak{so}_4 = \mathfrak{so}_3 \oplus \mathfrak{so}_3 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Lecture 9

Feb. 11 — Real Forms

9.1 Automorphisms of Semisimple Lie Algebras

Remark. Recall that we can identify $\mathrm{Aut}(\mathfrak{g})$ with a Lie group with Lie algebra $\mathfrak{g}$. The connected component of the identity $\mathrm{Aut}^\circ(\mathfrak{g})$ (also known as the *adjoint group* G_{ad}) acts transitively on the set of Cartan subalgebras. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then there is a connected subgroup $H \subseteq G_{\mathrm{ad}}$ which acts as 1 on $\mathfrak{h}$ and as $e^{\alpha(x)}$ on g_α for $x \in \mathfrak{h}$. Then we have

$$\mathfrak{h}/2\pi i P^\vee \cong H,$$

and H is called a *maximal torus.*

Proposition 9.1. *The normalizer $N(H)$ of H in G_{ad} coincides with the stabilizer of $\mathfrak{h}$ and contains H as a normal subgroup such that $N(H)/H = W$ (the Weyl group).*

Proof. Note that $\mathrm{SL}_2(\mathbb{C})$ is simply connected, so for any simple root α_i there is a homomorphism

$$\eta_i : \mathrm{SL}_2(\mathbb{C}) \longrightarrow G_{\mathrm{ad}} = \widetilde{G}/\mathcal{Z}(\widetilde{G}).$$

Define $S_i = \eta_i\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Consider $w = s_{i_1} \cdots s_{i_n}$ and $\widetilde{w} = S_{i_1} \cdots S_{i_n}$. Note that $\widetilde{w}$ acts on $\mathfrak{h}$ acts w, and if $w = w_1 w_2$, then $\widetilde{w} = \widetilde{w}_1 \widetilde{w}_2 h$ for some $h \in H$. To see the latter claim, note that h has the preserve the root decomposition, hence $h|_{\mathfrak{g}_{\alpha_j}} = \exp(b_j)$. Thus $h = \exp(\sum_j b_j w_j^\vee) \in H$ (where $\langle w_j^\vee, \alpha_i \rangle = \delta_{j,i}$).

So $\widetilde{w}$ and H generate a subgroup $N \subseteq N(H)$ such that $N/H = W$. It remains to show that $N(H) = N$. Let $x \in N(H)$ and consider simple roots $\alpha_i' = x(\alpha_i)$. Then there exists $w \in W$ such that $w(\alpha_i') = \alpha_{p(i)}$ for some permutation p. Then $\widetilde{w}x(\alpha_i) = \alpha_{p(i)}$, so this is a Dynkin diagram automorphism. Now $\widetilde{\omega}x$ is an element of a group, so the fundamental weights are fixed. Thus $p = \mathrm{id}$. $\square$

Remark. Although $N(H)/H = W$, in general $N(H)$ is not a semidirect product of H and W.

Proposition 9.2. *The map $\xi : \mathrm{Aut}(D) \ltimes G_{\mathrm{d}} \to \mathrm{Aut}(\mathfrak{g})$ is an isomorphism.*

Proof. We have to show that ξ is surjective. Let $a \in \mathrm{Aut}(\mathfrak{g})$. We can say that a preserves the Cartan subalgebra (if not, we can shift it by $g \in G_{\mathrm{ad}}$). Multiplying by $\mathrm{Aut}(D)N(H)$, we can make it act trivially on $\mathfrak{h}$ and $\mathfrak{g}_{\alpha_i}$. Then $a = 1$, so $a \in \operatorname{Im} \xi$. $\square$

9.2 Real Forms of Semisimple Lie Algebras

Remark. Recall the Serre presentation for $\mathfrak{g}$, i.e. generators $\{h_i, f_i, e_i\}$ with certain relations. In this setting, everything was defined over $\mathbb{Q}$.

Definition 9.1. A semisimple Lie algebra is *split* if it admits a Chevalley-Serre basis over base field K.

Remark. Let L be a Galois extension of K ($\operatorname{char} K = 0$), and assume that $\mathfrak{g}_L$ is a split semisimple Lie algebra. We want to find $\mathfrak{g}$ over K such that $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$. The problem is then to find a classification of all such $\mathfrak{g}$. Let $\Gamma = \mathrm{Gal}(L/K)$. Define an action of Γ on $\mathfrak{g}_L$ by

$$g(\lambda x) = g(\lambda) g(x), \quad x \in \mathfrak{g}_L, \lambda \in L, g \in \Gamma,$$

which is twisted linear. We can reconstruct $\mathfrak{g}$ as the invariants $\mathfrak{g}_L^\Gamma$.

The simplest action of this kind is $\rho_0(g)$, which acts on scalars and preserves $\{h_i, e_i, f_i\}$. Any twisted linear action takes the form $\rho(g) = \eta(g)\rho_0(g)$ for some $\eta : \Gamma \to \mathrm{Aut}(\mathfrak{g}_L)$. As ρ is a homomorphism,

$$\eta(gh)\rho_0(gh) = \eta(g)\rho_0(g)\eta(h)\rho_0(h),$$

and upon rearranging, we have

$$\eta(gh) = \eta(g) g(\eta(h)),$$

where $g(a) = \rho_0(g) a \rho_0(g)^{-1}$ for $a \in \mathrm{Aut}(\mathfrak{g}_L)$. The above is called a 1-*cocycle condition*.

Denote the Lie algebra associated to the cocycle η by $\mathfrak{g}_\eta$. When do we have $\mathfrak{g}_{\eta_1} \cong \mathfrak{g}_{\eta_2}$? This is the case when ρ_1 and ρ_2 are isomorphic, i.e. there exists $a \in \mathrm{Aut}(\mathfrak{g}_L)$ such that $\rho_1(g)a = a\rho_2(g)$. Then

$$\eta_1(g)\rho_0(g)a = a\eta_2(g)\rho_0(g),$$

so $\eta_1(g) = a\eta_2(g)g(a)^{-1}$. Thus η_1 and η_2 are cohomologous cocycles.

Proposition 9.3. *The semisimple Lie algebras $\mathfrak{g}$ over K which split over L (where L/K is Galois) are classified by $H^1(\Gamma, \mathrm{Aut}(\mathfrak{g}_L))$, where $\Gamma = \mathrm{Gal}(L/K)$.*

Remark. We will now specialize to $K = \mathbb{R}$, $L = \mathbb{C}$, where $\Gamma = \mathbb{Z}/2\mathbb{Z}$, generated by complex conjugation. We have $\mathrm{Aut}(\mathfrak{g}_L) = \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$. Since $\eta(1) = 1$, η is determined by $\eta(-1)$. The cocycle condition is

$$s\bar{s} = 1, \quad s = \eta(-1).$$

The corresponding Lie algebra (up to isomorphism) depends only on the cohomology class of s, where $s \to as\bar{a}^{-1}$ for $a \in \mathrm{Aut}(D)$.

Theorem 9.1. *The real semisimple Lie algebras whose complexification is $\mathfrak{g}$ (i.e. the real forms of $\mathfrak{g}$) are classified by $s \in \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$ such that $s\bar{s} = 1$, modulo the equivalence $s \to as\bar{a}^{-1}$ for $a \in \mathrm{Aut}(D)$.*

Remark. Note that complex conjugation acts trivially on $\mathrm{Aut}(D)$.

Remark. Denote by $\mathfrak{g}_{(s)} = \{x \in \mathfrak{g} : \bar{x} = s(x)\}$ the real form corresponding to s. Denote by $\mathfrak{g}_{(1)}$ the split form consisting of real $x \in \mathfrak{g}$ (so that $x = \bar{x}$).

Alternatively, we can define an antilinear involution $\sigma_s(x) = \overline{s(x)}$. Then $\mathfrak{g}_{(s)}$ is the fixed point set of σ_s.

Remark. Note that s defines $s_0 \in \mathrm{Aut}(D)$ with $s_0^2 = 1$.

Corollary 9.1.1. *The conjugacy class of s_0 is invariant under equivalences.*

Remark. Since s_0 permutes the connected components of the Dynkin diagram D, it preserves some and divides some into pairs. So every semisimple real Lie algebra is a direct sum of simple ones, and each simple one has either connected Dynkin diagram or consists of two identical components.

Remark. We now consider the case when D is connected and $\mathfrak{g}$ is simple.

Definition 9.2. A real form $\mathfrak{g}_{(s)}$ of a complex simple Lie algebra is said to be *inner to* $\mathfrak{g}_{(s')}$ if $s' = gs$ up to equivalence, where $g \in G_{\text{ad}}$ (i.e. s, s' differ by an inner automorphism). The *inner class* of $\mathfrak{g}_{(s)}$ is the collection of all real forms inner to $\mathfrak{g}_{(s)}$. An *inner form* is a form inner to a split form. We call $\mathfrak{g}_{(s)}$ *quasi-split* if $s = s_0 \in \operatorname{Aut}(D)$.

Corollary 9.1.2. *We have the following:*

1. *Any real form is inner to a unique quasi-split form.*
2. *A real form which is both inner and quasi-split is split.*

Example 9.2.1. Consider the *Cartan involution* τ defined by

$$\tau(h_j) = -h_j, \quad \tau(e_j) = -f_j, \quad \tau(f_j) = -e_j.$$

Then $\mathfrak{g}_{(\tau)} = \mathfrak{g}^c$ is called the *compact real form* of $\mathfrak{g}$.

Lecture 10

Feb. 16 — Real Forms, Part 2

10.1 Compact Real Forms

Proposition 10.1. *Let τ be the Cartan involution (defined in Example 9.2.1) and $\mathfrak{g}^c = \mathfrak{g}_{(\tau)}$. Then the Killing form of $\mathfrak{g}^c$ is negative definite.*

Proof. We can write $\mathfrak{g}^c = (\mathfrak{h} \cap \mathfrak{g}^c) \oplus \bigoplus_{\alpha \in R_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c$. The Killing form is negative definite on $\mathfrak{h} \cap \mathfrak{g}^c$ since the inner product on a coroot lattice is positive definite. Thus it is negative definite on $\mathfrak{g}^c$, as $\{i\alpha_j^\vee\}$ is a basis for $\mathfrak{g}^c \cap \mathfrak{h}$. Now we need to show that it is negative definite on $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c$. Note that for $\mathfrak{g} = \mathfrak{sl}_2$, we have a basis for $\mathfrak{g}^c$ given by ih, $e - f$, $i(e+f)$, so $\mathfrak{g}^c = \mathfrak{su}(2)$. So the statement holds there. For general $\mathfrak{g}$, we have that S_i preserves $\mathfrak{g}^c$, since

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2)$$

and $\mathrm{Lie}(\mathrm{SU}(2)_i) \subseteq \mathfrak{g}^c$. Thus for any $w \in W$, the lift $\widetilde{w}$ preserves $\mathfrak{g}^c$, so the restriction of the Killing form of $\mathfrak{g}^c$ to $\mathfrak{g}^c \cap (\mathfrak{sl}_2)_\alpha$ is negative definite. ☐

Remark. Consider $\mathrm{Aut}(\mathfrak{g}^c)$. Since the Killing form is negative definite, $\mathrm{Aut}(\mathfrak{g}^c)$ is a closed subgroup of $\mathrm{O}(\mathfrak{g}^c)$, so it is compact. Moreover, this is a Lie group with Lie algebra $\mathfrak{g}^c$.

Corollary 10.0.1. *$G^c_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g}^c)^\circ$ is a connected, compact Lie group with Lie algebra $\mathfrak{g}^c$.*

Example 10.0.1. We have the following:

1. For $\mathfrak{g} = \mathfrak{sl}_n$, we have $G^c_{\mathrm{ad}} = \mathrm{PSU}(n) = \mathrm{SU}(n)/\mu_n$, where μ_n is the nth roots of unity.
2. For $\mathfrak{g} = \mathfrak{so}_n$, we have $G^c_{\mathrm{ad}} = \mathrm{SO}(n)$ for odd n and $G^c_{\mathrm{ad}} = \mathrm{SO}(n)/\{\pm 1\}$ for even n.
3. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $G^c_{\mathrm{ad}} = \mathrm{U}(n, \mathbb{H})/\{\pm 1\}$. The group

$$\mathrm{U}(n, \mathbb{H}) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{U}(2n)$$

is called the *quaternionic unitary group.*

Example 10.0.2. We have the following:

1. Consider A_{n-1}, which corresponds to the split form $\mathfrak{sl}(n, \mathbb{R})$ and compact form $\mathfrak{su}(n)$. For $n > 2$, we have a quasi-split real form as follows: Let $s(A) = -JA^T J^{-1}$ where $J_{i,j} = (-1)^i \delta_{i,n+1-j}$. Then

$$e_i, f_i, h_i, \longmapsto e_{n+1-i}, f_{n+1-i}, h_{n+1-i}.$$

Note that J is a Hermitian or skew-Hermitian form of signature (p, p) with $n = 2p$ or of signature $(p+1, p), (p, p+1)$, which are isomorphic when $n = 2p + 1$.

For $n = 2$, we have $\mathfrak{su}(1,1) = \mathfrak{sl}(2, \mathbb{R})$ (and $\mathrm{PSU}(1,1) = \mathrm{PSL}(2, \mathbb{R})$).[1]

2. For type B_n, we have compact form $\mathfrak{so}(2n+1)$ and split form $\mathfrak{so}(n+1, n)$. There are no nontrivial automorphisms, so there are no non-split quasi-split forms.

   Particular cases of interest are $\mathrm{SO}(3) \cong \mathrm{SU}(2)$ and $\mathrm{SO}^+(2,1) = \mathrm{PSU}(1,1) = \mathrm{PSL}(2, \mathbb{R})$.

3. For type C_n, we have the split form $\mathfrak{sp}(2n, \mathbb{R})$ and compact form $\mathfrak{u}(n, \mathbb{H})$. There are no non-split quasi-split reals forms, as there are no nontrivial automorphisms of the Dynkin diagram.

   Note that $B_2 = C_2$, so we have $\mathfrak{so}(3,2) = \mathfrak{sp}_4(\mathbb{R})$ and $\mathfrak{so}(5) = \mathfrak{u}(2, \mathbb{H})$.

4. For type D_n, we have split form $\mathfrak{so}(n,n)$ and compact form $\mathfrak{so}(2n)$. For $n > 4$, there is a unique nontrivial involution, while for $n = 4$, we have $\mathrm{Aut}(D) = S_3$. However, there is still a unique non-split quasi-split form as there is only one nontrivial involution up to conjugation. Recall

   $$A = -JA^TJ^{-1}, \quad J_{i,j} = \delta_{i,2n+1-j}.$$

   Then the quasi-split form is given by $J \mapsto J' = gJ$, where g permutes e_n, e_{n+1} (which corresponds to α_{n-1}, α_n). The signature defined by J' is $(n+1, n-1)$, so the quasi-split form is $\mathfrak{so}(n+1, n-1)$.

   Note that $D_2 = A_1 \oplus A_1$, so we have the following isomorphisms:

   $$\begin{aligned} \mathfrak{so}(4) &= \mathfrak{su}(2) \oplus \mathfrak{su}(2), \\ \mathfrak{so}(2,2) &= \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1), \\ \mathfrak{so}(3,1) &= \mathfrak{sl}_2(\mathbb{C}), \end{aligned}$$

   where in the last isomorphism we view $\mathfrak{sl}_2(\mathbb{C})$ as a real Lie algebra.

   We also have $D_3 = A_3$, which gives the following isomorphisms:

   $$\begin{aligned} \mathfrak{so}(6) &= \mathfrak{su}(4), \\ \mathfrak{so}(3,3) &= \mathfrak{sl}_4(\mathbb{R}), \\ \mathfrak{so}(4,2) &= \mathfrak{su}(2,2). \end{aligned}$$

10.2 Classification of Real Forms

Remark. Write $\mathfrak{g} = \mathfrak{g}^c \otimes_{\mathbb{R}} \mathbb{C}$ and $\omega = \sigma_\tau$ the Cartan antilinear involution (so that $\mathfrak{g}^c$ is the fixed points of σ_τ). Another real structure on $\mathfrak{g}$ is given by $\sigma = \omega \circ g$ for $g \in \mathrm{Aut}(\mathfrak{g})$, as

$$\sigma^2 = \omega \circ g \circ \omega \circ g = 1$$

(note that $\omega \circ g \circ \omega = \omega(g)$, and $\omega(g)g = 1$). Define

$$(X, Y) = \mathrm{tr}(\mathrm{ad}_X \, \mathrm{ad}_{\omega(Y)}),$$

which is the Hermitian extension of the Killing form from $\mathfrak{g}^c$ to $\mathfrak{g}$. Note that $\omega(g) = (g^\dagger)^{-1}$, where $g^\dagger$ is the adjoint of g. Thus we see that g is self-adjoint.

[1] Note that $\mathrm{PSU}(1,1)$ is the group of automorphisms of the unit disk, and $\mathrm{PSL}(2, \mathbb{R})$ is the group of automorphisms of the upper half-plane. The isomorphism comes from the Cayley transform from the unit disk to the upper half-plane.

In particular, g is diagonalizable with real eigenvalues. So we can write

$$\mathfrak{g} = \bigoplus_{\gamma \in \mathbb{R}} \mathfrak{g}(\gamma),$$

where $\mathfrak{g}(\gamma)$ is the eigenspace of $\mathfrak{g}$ corresponding to eigenvalue γ. Note that

$$[\mathfrak{g}(\beta), \mathfrak{g}(\gamma)] = \mathfrak{g}(\beta\gamma).$$

Consider the operator $|g|^t$ for $t \in \mathbb{R}$, which acts on $\mathfrak{g}(\gamma)$ as $|\gamma|^t$. We can rewrite

$$|g|^t = \exp(t \log |g|) \in G_{\mathrm{ad}},$$

which is a 1-parameter subgroup of G_{ad}. Then we can define $\theta := g|g|^{-1}$, and we have

$$\begin{cases} \theta \circ \omega = \omega \circ \theta, \\ \theta^2 = 1, \end{cases}$$

where the first identity follows from $(\theta^\dagger)^{-1} = \theta$. We have

$$\theta = |g|^{-1/2} g \omega(|g|^{1/2}).$$

We can assume $g = \theta$ with $\theta \circ \omega = \omega \circ \theta$, or equivalently, that $\theta \in \mathrm{Aut}(\mathfrak{g}^c)$ and $\theta^2 = 1$. Thus θ has ± 1 eigenspaces. Note that θ' defines the same real form if and only if

$$\theta' = x\theta(\omega(x))^{-1}$$

for some $x \in \mathrm{Aut}(\mathfrak{g})$. Then we have $x\theta(\omega(x))^{-1} = \omega(x)\theta x^{-1}$ (since $\theta'^2 = 1$, so $\theta'^{-1} = \theta'$). Let

$$z = (\omega(x))^{-1} x,$$

so that $\omega(z) = z^{-1}$. Then $\theta z = z^{-1}\theta$. Now note that $z = x^\dagger x$ is positive definite, so if $y = xz^{-1/2}$,

$$\omega(y) = \omega(x) z^{1/2} = (x^\dagger)^{-1} z^{1/2} = xz^{-1/2} = y.$$

Thus $y \in \mathrm{Aut}(\mathfrak{g}^c)$, and we also have that

$$\theta' = x\theta\omega(x)^{-1} = x\theta z x^{-1} = xz^{-1/2}\theta z^{1/2} x^{-1} = y\theta y^{-1}.$$

Theorem 10.1. *The real forms of $\mathfrak{g}$ are in one-to-one correspondence with the conjugacy classes of involutions $\theta \in \mathrm{Aut}(\mathfrak{g}^c)$, where $\theta \mapsto \omega_\theta := \theta \circ \omega = \omega \circ \theta$.*

Remark. For $\theta : \mathfrak{g} \to \mathfrak{g}$, denote by $\mathfrak{g}_\theta$ the corresponding real form. For example,

$$\mathfrak{g}_1 = \mathfrak{g}^c = \mathfrak{g}_{(\tau)},$$

where the latter is our old notation using the split forms.

Remark. We now have a canonical (up to automorphisms of $\mathfrak{g}^c$) decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\theta = 1$ on $\mathfrak{k}$ and $\theta = -1$ on $\mathfrak{p}$. Here $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. For $\mathfrak{g}^c$ itself, we have

$$\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c.$$

Moreover, we have $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$, where $\mathfrak{p}_\theta = i\mathfrak{p}^c$ (a fixed point of $\sigma = \omega \circ \theta$ has to have an extra $-$ sign, which we can achieve by multiplying by i).

Exercise 10.1. Show that $\mathfrak{k}$ is reductive but not necessarily semisimple.

Lecture 11

Feb. 18 — Real Forms, Part 3

11.1 Classification of Real Forms, Continued

Proposition 11.1. *There exists a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ which is invariant under θ and such that $\mathfrak{h} \cap \mathfrak{k}$ is a Cartan subalgebra in $\mathfrak{k}$.*

Proof. Consider a generic element $t \in \mathfrak{k}^c$. It is regular and semisimple. Consider $\mathfrak{h}_+^c$, the centralizer of t in $\mathfrak{k}^c$. Then necessarily $\mathfrak{h}_+ := \mathfrak{h}_+^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra in $\mathfrak{k}$. Let $\mathfrak{h}_-^c$ be a maximal subspace of $\mathfrak{p}^c$ so that $\mathfrak{h}^c = \mathfrak{h}_+^c \oplus \mathfrak{h}_-^c$ is a commutative subalgebra of $\mathfrak{g}^c$. Then we claim that $\mathfrak{h} = \mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$. Note that $\mathfrak{h}$ consists of the semisimple elements, and all elements in $\mathfrak{g}^c$ are anti-self-adjoint operators. If $z \in \mathfrak{g}$ commutes with $\mathfrak{h}$, then

$$z = z_+ + z_-, \quad z_+ \in \mathfrak{k}, z_- \in \mathfrak{p},$$

where $z_\pm$ commute with $\mathfrak{h}$. Then $z_+ \in \mathfrak{h}_+$, and

$$z_- = x + iy, \quad x, y \in \mathfrak{p}^c,$$

where x, y commute with $\mathfrak{h}$. Then $x, y \in \mathfrak{h}_-^c$ (by the definition of $\mathfrak{h}_-^c$), so $z \in \mathfrak{h}$. $\square$

Corollary 11.0.1. *We have $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$, where $\theta = 1$ on $\mathfrak{h}_+$ and $\theta = -1$ on $\mathfrak{h}_-$.*

Lemma 11.1. *There are no coroots of $\mathfrak{g}$ in $\mathfrak{h}_-$.*

Proof. Suppose otherwise that $\alpha^\vee \in \mathfrak{h}_-$. Then $\theta(\alpha^\vee) = -\alpha^\vee$, so

$$\theta(e_\alpha) = e_{-\alpha} \quad \text{and} \quad \theta(e_{-\alpha}) = e_\alpha$$

for some $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$. Then $x = e_\alpha + e_{-\alpha}$ satisfies $\theta(x) = x$, so $x \in \mathfrak{k}$. But $x \notin \mathfrak{h}_+$ (since $x \perp \mathfrak{h}_+$). Thus $[\mathfrak{h}_+, x] = 0$ since α vanishes on $\mathfrak{h}_+$, a contradiction as $\mathfrak{h}_+$ is a maximal commutative subalgebra of $\mathfrak{k}$. $\square$

Remark. Pick a generic element $t \in \mathfrak{h}_+$, which is regular in $\mathfrak{g}$. Choose t so that

$$\mathrm{Re}(t, \alpha^\vee) \neq 0$$

for all $\alpha^\vee$ of $\mathfrak{g}$. Then we can define a *polarization* on R by

$$R_+ = \{\alpha \in R : \mathrm{Re}(t, \alpha^\vee) > 0\}$$

which satisfies $\theta(R_+) = R_+$. Now $\{\theta(i) : i \in D\}$ gives the action of θ: If $\theta = i$, then

$$\theta(e_i) = \pm e_i, \quad \theta(h_i) = h_i, \quad \theta(f_i) = \pm f_i.$$

Otherwise, if $\theta(i) \neq i$, then we can choose generators $h_i, e_i, e_{\theta(i)}, f_i, f_{\theta(i)}, h_{\theta(i)}$ such that

$$\theta(x_i) = x_{\theta(i)}, \quad x = e, f, h.$$

We can then construct *markings* on the Dynkin diagram as follows:

- Connect vertices i and $\theta(i)$ if $\theta(i) \neq i$.
- Mark a vertex i as white if $\theta(e_i) = e_i$.
- Mark a vertex i as black if $\theta(e_i) = -e_i$.

This is the *Vogan diagram* associated to the Dynkin diagram. Note that $e_i \in P$ is a non-compact root.

Exercise 11.1. Showing the following:

1. The signature of the Killing form of g_θ is $(\dim \mathfrak{p}, \dim \mathfrak{k})$. Moreover, the Killing form is negative definite if and only if $\theta = 1$, i.e. $\mathfrak{g} = \mathfrak{g}^c$.
2. For a split real form, $\dim \mathfrak{k} = |R_+|$.
3. Show that for any real form in a compact inner class, $\operatorname{rank}(\mathfrak{k}) = \operatorname{rank}(\mathfrak{g})$.

11.2 Real Forms of Classical Lie Algebras

Example 11.0.1. We have the following real forms of the classical Lie algebras:

1. Type A_{n-1}, compact inner class.

   Let θ be the inner automorphism element of $\mathrm{PSU}(n)$ of order 2. Let $g \in \mathrm{U}(n)$ such that $g^2 = 1$. Then $\theta(x) = gxg^{-1}$, so $g = \mathrm{id}_p \otimes (-\mathrm{id}_q)$ with $p + q = n$. Thus

   $$\mathfrak{g}_\theta = \mathfrak{su}(p, q) \quad \text{and} \quad \mathfrak{k} = \mathfrak{gl}_p \oplus \mathfrak{sl}_q.$$

   For $n = 2$, we have $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$, with $\mathfrak{k} = \mathfrak{gl}_1$.

2. Type A_{n-1}, split inner class.

   If n is odd (so all vertices are divided into connected pairs), then

   $$\mathfrak{g}_\theta = \mathfrak{sl}_n(\mathbb{R}).$$

   If n is even, then there is 1 stable vertex (which is either black or white). In these cases we either have $\mathfrak{k} = \mathfrak{sp}_{2k}$ (in which case $\mathfrak{g}_\theta = \mathfrak{sl}(k, \mathbb{H})$) or $\mathfrak{k} = \mathfrak{so}_{2k}$ (which is just the split form $\mathfrak{sl}_n(\mathbb{R})$).

3. Type B_n (i.e. $\mathfrak{so}_{2n+1}$).

   Let θ be the inner automorphism of order ≤ 2. We can write

   $$\theta = \mathrm{id}_{2p+1} \oplus (-\mathrm{id}_{2q})$$

   where $p + q = n$. The real forms are $\mathfrak{so}(2p+1, 2q)$, with $\mathfrak{k} = \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2q}$.

4. Type C_n.

   We can have $g \in \mathrm{Sp}_{2n}(\mathbb{C})$ such that $g^2 = 1$ or $g^2 = -1$. The adjoint compact group is

   $$(\mathrm{Sp}(2n) \cap U(n))/\pm 1.$$

If $g^2 = 1$, then the eigenspace with eigenvalue 1 has dimension $2p$, and the eigenspace for -1 has dimension $2q$ (where $p + q = n$). Assume that $p \geq q$ (otherwise take $g \mapsto -g$). Then we have

$$\mathfrak{sp}(2p, 2q) = \mathfrak{sp}_{2n} \cap \mathfrak{u}(p, q) = \mathfrak{u}(p, q, \mathbb{H}).$$

In this case, we have $\mathfrak{k} = \mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2q}$.

If $g^2 = -1$, then $\mathbb{C}^{2n} = V(i) \oplus V(-i)$. In this case, $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$, as for any $(w, w) = w \cdot \overline{w}$ in $\mathbb{C}^n$,

$$\mathrm{Im}(w, w) = i\,\mathrm{Im}(w \wedge \overline{w})$$

defines a symplectic form on $\mathbb{R}^{2n}$. Thus we can view $\mathrm{U}(n) \subseteq \mathrm{Sp}_{2n}(\mathbb{R})$.

5. Type D_n, compact inner class.

   In this case, θ is given by $g \in \mathrm{SO}(2n)$ with $g^2 = \pm 1$.

   If $g^2 = 1$, then $\mathbb{C}^{2n} = V(1) \oplus V(-1)$ (where $\dim V(1) = 2p$ and $\dim V(-1) = 2q$ with $p + q = n$). Note that $\det(g) = 1$, so the eigenspaces are even-dimensional. Then

   $$\mathfrak{g}_\theta = \mathfrak{so}(2p, 2q) \quad \text{and} \quad \mathfrak{k} = \mathfrak{so}(2p) \oplus \mathfrak{so}(2q).$$

   If $g^2 = -1$, then $\mathbb{C}^{2n} = V(i) \oplus V(-i)$, so $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$. In this case, we have

   $$\mathfrak{g}_\theta = \mathfrak{so}^*(2n),$$

   which is known as the *quaternionic orthogonal Lie algebra*.

6. Type D_n, the other inner class.

   In this case, θ is given by $g \in \mathrm{O}(2n)$ with $\det(g) = -1$ and $g^2 = \pm 1$.

   Note that if $g^2 = -1$, then $\det(g) = 1$, which is a contradiction. So we can only have $g^2 = 1$. Then

   $$\mathbb{C}^{2n} = V(1) \oplus V(-1),$$

   where $\dim V(1) = 2p + 1$ and $\dim V(-1) = 2q + 1$. Here $\mathfrak{k} = \mathfrak{so}(2p + 1) \oplus \mathfrak{so}(2q + 1)$.

11.3 More on Compact Groups

Exercise 11.2. Show that if K^c is a compact Lie group, then $\mathfrak{k} = \mathrm{Lie}_{\mathbb{C}}(K^c)$ is a reductive Lie algebra.

Example 11.0.2. Let $G_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g})^\circ$ for a semisimple Lie algebra $\mathfrak{g}$, and let G_{ad}^c be its compact form. Consider the following product:

$$(S^1)^r \times G_{\mathrm{ad}}^c.$$

We will see that any Lie algebra of a compact group is isomorphic to a Lie algebra of such a product. We can also consider covering spaces, i.e. what is $\pi_1(G_{\mathrm{ad}}^c)$?

Lecture 12

Feb. 23 — Classifications of Lie Groups

12.1 Classification of Compact Lie Groups

Remark. Let $\mathfrak{g}$ be semisimple and G the corresponding simply connected Lie group (which is the universal cover for $G_{\text{ad}} = \text{Aut}(\mathfrak{g})^{\circ}$). Define

$$Z = \ker(G \to G_{\text{ad}}) = \pi_1(G_{\text{ad}}).$$

Recall that the finite-dimensional representations of G are in one-to-one correponsdence with the finite-dimensional representations of $\mathfrak{g}$, which (the irreducible ones) are given by $\{L_\lambda\}$ for $\lambda \in P_+$. Then Z acts as $\chi_\lambda : Z \to \mathbb{C}^\times$ on every L_λ, and $\chi_\lambda \chi_\mu = \chi_{\lambda+\mu}$. Note that

$$\chi : P \longrightarrow \text{Hom}(Z, \mathbb{C}^\times),$$

and $\chi_\theta = 1$ (where θ is the longest root) as Z acts trivially on the adjoint representation.

Exercise 12.1. If $\lambda(h_i)$ is sufficiently large, then show that for all $\alpha \in \mathfrak{g}$ we have

$$L_{\lambda+\alpha} \subseteq L_\lambda \otimes \mathfrak{g},$$

so in particular, $\chi_{\lambda+\alpha} = \chi_\lambda$, i.e. $\chi_\alpha = 1$.

Remark. We can define maps $P/Q \to \text{Hom}(Z, \mathbb{C}^\times)$ and $Z \to P^\vee/Q^\vee$ (and similarly for $G^c_{\text{ad}}, G^c, Z^c$).

Proposition 12.1. *A representation L_λ of $\mathfrak{g}$ of highest weight $\lambda \in P_+$ lifts to a representation of G_{ad} (equivalently, of G^c_{ad}) if and only if $\lambda \in P_+ \cap Q$.*

Proof. We have shown if $\lambda \in P_+ \cap Q$, then L_λ lifts. The converse follows from Proposition 12.2. □

Proposition 12.2. *If V is a faithful finite-dimensional representation of a compact Lie group, then any irrep Y is contained in $V^{\otimes n} \otimes (V^*)^{\otimes m}$ for some n, m.*

Lemma 12.1. *If X is a compact manifold, then $\pi_1(X)$ is finitely generated.*

Proof. Cover X with balls around each point, by the compactness of X there is a finite subcover. Let $x_1, \ldots, x_n$ be the centers of the finitely many balls, and create a graph G by connecting x_i. Then there is a surjection $\pi_1(G) \to \pi_1(X)$, and $\pi_1(G)$ is finitely generated. □

Theorem 12.1. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and G^c_{ad} the corresponding adjoint compact group. Then $\pi_1(G^c_{\text{ad}}) = P^\vee/Q^\vee$. In particular, the universal cover of G^c_{ad} is a compact Lie group.*

Proof. Let G_*^c be a finite cover of G_{ad}^c and define

$$Z_{G_{\mathrm{ad}}^c} = \ker(G_*^c \to G_{\mathrm{ad}}^c) \subseteq G_*^c.$$

A finite-dimensional irrep is classified by $P_+(G_*^c) \subseteq P_+$ with $P_+ \cap Q \subseteq P_+(G_*^c)$. Let $P(G_*^c) \subseteq P$ be the lattice generated by $P_+(G_*^c)$, and consider the character χ_λ for the action of $Z_{G_{\mathrm{ad}}^c}$ on L_λ (an irrep of G_*^c). Then χ gives a map

$$\xi : P(G_*^c)/Q \longrightarrow Z_{G_{\mathrm{ad}}^c}^\vee = \mathrm{Hom}(Z_{G_{\mathrm{ad}}^c}, \mathbb{C}^\times).$$

Since G_*^c is compact, by the Peter-Weyl theorem ξ is surjective. It just remains to show that $\pi_1(G_{\mathrm{ad}}^c)$ is finite. Let $G_*^c = G^c$, the universal cover. Then $P(G_*^c) = P$, so $P/Q \cong Z^\vee$, and thus

$$Z = \pi_1(G_{\mathrm{ad}}^c) \cong P^\vee/Q^\vee.$$

By Lemma 12.1, $\pi_1(G_{\mathrm{ad}}^c)$ is finitely generated, and it is also abelian. Take a subgroup of finite index N and take G_*^c to be the corresponding cover. Then we have

$$N = |Z_{G_*^c}| \leq |P(G_*^c)/Q| \leq |P/Q|,$$

which for abelian groups implies that the group is finite. □

Corollary 12.1.1. *We have the following:*

1. *If $\mathfrak{g}$ is a semisimple complex Lie algebra, then the simply connected Lie group G^c corresponding to the Lie algebra $\mathfrak{g}^c$ is compact and its center is $P^\vee/Q^\vee$, which is the same as $\pi_1(G_{\mathrm{ad}}^c)$.*
2. *Let $\Gamma \subseteq P^\vee/Q^\vee$. Then the irreps of G/Γ are the L_λ such that λ defines the trivial character of Γ.*
3. *Let G_i^c be compact Lie groups corresponding to simple Lie algebras $\mathfrak{g}_i$, and let $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$. Then any connected Lie group with Lie algebra $\mathfrak{g}^c$ is compact and of the form*

$$\left(\prod_{i=1}^n G_i^c\right)/Z,$$

where $Z = \pi_1(G^c)$ is a subgroup of $\prod_i Z_i$ for $Z_i = P_i^\vee/Q_i^\vee$ (the center of G_i^c). Moreover, every semisimple connected compact Lie group has this form.

Example 12.0.1. Let $G_*^c = \mathrm{SO}(4,\mathbb{R})$. Then $G_{\mathrm{ad}}^c = \mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$ and $G^c = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and

$$\mathrm{SO}(4,\mathbb{R}) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(1,1)\}.$$

Example 12.0.2. Let $G_* = \mathrm{SO}(4,\mathbb{C})$. Then $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) = \mathrm{Spin}(4)$, and

$$\mathrm{SO}(4,\mathbb{C}) = (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))/\{\pm(1,1)\}.$$

In this case, $G_{\mathrm{ad}} = \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$.

Corollary 12.1.2. *Any connected compact Lie group with Lie algebra of the form $\mathfrak{g}^c \oplus \mathfrak{a}$ with $\mathfrak{a}$ abelian is a quotient of $T \times C$ by a finite central subgroup, where $T = (S^1)^m$ and C is compact, semisimple, and simply connected.*

12.2 Polar Decomposition

Remark. Consider $G_{\mathrm{ad},\theta} \subseteq G_{\mathrm{ad}}$ corresponding to $\mathfrak{g}_\theta \subseteq \mathfrak{g}$. Note that $G_{\mathrm{ad},\theta}$ is a closed subgroup, but it may be disconnected (e.g. if $\mathfrak{g}_\theta = \mathfrak{sl}(2,\mathbb{R}) \subseteq \mathfrak{sl}(2,\mathbb{C})$, then $G_{\mathrm{ad}} = \mathrm{PGL}_2(\mathbb{C})$ and $G_{\mathrm{ad},\theta} = \mathrm{PGL}_2(\mathbb{R})$, which is disconnected as $\det : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$ and $\mathbb{R} \setminus \{0\}$ is disconnected; but $\mathbb{C} \setminus \{0\}$ is connected).

Now let $K^c \subseteq G_{\mathrm{ad},\theta}$ be the subgroup of elements acting on $\mathfrak{g}$ by unitary operators, i.e. K^c is the fixed points of ω_θ. Note that K^c is a closed but possibly disconnected subgroup. Let $\mathrm{Lie}(K^c) = \mathfrak{k}^c$, and note that K^c is compact. Let $P_\theta = \exp(\mathfrak{p}_\theta) \subseteq G_{\mathrm{ad},\theta}$, where $\mathfrak{p}_\theta = i\mathfrak{p}^c$ (note that P_θ need not be a subgroup). Now P_θ acts on $\mathfrak{g}$ by Hermitian operators, so we get a diffeomorphism

$$\exp : \mathfrak{p}_\theta \to P_\theta,$$

thus P_θ is a closed embedded submanifold in $G_{\mathrm{ad},\theta}$.

Theorem 12.2 (Polar decomposition)**.** *The multiplication $K^c \times P_\theta \to G_{\mathrm{ad},\theta}$ is a diffeomorphism, hence*

$$G_{\mathrm{ad},\theta} \cong K^c \times \mathbb{R}^{\dim \mathfrak{p}}$$

as manifolds. In particular, $G_{\mathrm{ad},\theta}$ is homotopy equivalent to K^c.

Proof. By the polar decomposition for matrices, any invertible matrix A can be written as $A = U_A R_A$, where U_A is unitary and R_A is positive Hermitian. Explicitly, we have

$$R_A = (A^\dagger A)^{1/2} \quad \text{and} \quad U_A = A(A^\dagger A)^{-1/2}.$$

Let $g \in G_{\mathrm{ad},\theta} \subseteq \mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$. Note that $g^\dagger g$ is an automorphism of $\mathfrak{g}$ with positive eigenvalues, so $(g^\dagger g)^{1/2} = R_g \in P_\theta$, the positive self-adjoint elements. Since U_g is unitary, it has to belong to K^c. Thus we have constructed an inverse map to μ. $\square$

Corollary 12.2.1. *We have $G_{\mathrm{ad}} \cong G^{\mathrm{ad}} \times \mathbb{P}$, where $\mathbb{P}$ is the set of elements acting on $\mathfrak{g}$ by positive Hermitian operators. In particular, $\pi_1(G_{\mathrm{ad}}) = \pi_1(G^c_{\mathrm{ad}}) = P^\vee/Q^\vee$.*

Corollary 12.2.2. *If G is a semisimple complex Lie group, then the center Z of G is contained in G^c, hence Z coincides with the center Z^c of G^c. Thus the restriction of finite-dimensional representations from G to G^c defines an equivalence of categories.*

Remark. By considering coverings, the polar decomposition also applies to the real form $G_\theta \subseteq G$ of any connected complex semisimple Lie group G. However, note that G_θ need not be simply connected even if G is (e.g. for $G = \mathrm{SL}_2(\mathbb{C})$ we have $G_\theta = \mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SO}(2) \cong S^1$, so $\pi_1(G_\theta) \cong \mathbb{Z}$).

Example 12.0.3. We have the following:

1. If $G = \mathrm{SL}_n(\mathbb{C})$, then $K^c = \mathrm{SU}(n)$ and P is the set of positive Hermitian matrices of determinant 1.
2. If $G_\theta = \mathrm{SL}_n(\mathbb{R})$, then $K^c = \mathrm{SO}(n)$ and P_θ is the positive symmetric matrices of determinant 1.

Lecture 13

Feb. 25 — Maximal Tori

13.1 Classification of Reductive Lie Groups

Definition 13.1. A connected complex Lie group G is *reductive* if it has the form

$$((\mathbb{C}^\times)^r \times G_{\mathrm{ss}})/Z,$$

where G_{ss} is semisimple and Z is a finite central subgroup. A complex Lie group G is called *reductive* if G° is reductive and G/G_0 is finite.

Example 13.1.1. Recall that $\mathrm{GL}_n(\mathbb{C}) = (\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))/\mu_n$, hence $\mathrm{GL}_n(\mathbb{C})$ is reductive.

Remark. With this definition, a simply connected complex Lie group with reductive Lie algebra is not necessarily reductive (e.g. $\mathbb{C}$).

Remark. Given $G = ((\mathbb{C}^\times)^r \times G_{\mathrm{ss}})/Z$, we have $Z \subseteq (S^1)^r \times G_{\mathrm{ss}} \subseteq (\mathbb{C}^\times)^r \times G_{\mathrm{ss}}$. Then

$$G^c = ((S^1)^r \times G^c_{\mathrm{ss}})/Z \subseteq G.$$

Recall that restriction from G to G^c defines an equivalence of categories between their representations, so the representations of G are completely reducible and the irreps are characterized by

$$(n_1, \ldots, n_r, \lambda), \quad \lambda \in P_+(G_{\mathrm{ss}}),\ n_i \in \mathbb{Z}.$$

Thus we have the characterize the trivial character of Z.

Definition 13.2. A connected Lie group G is called *linear* if it can be realized as a subgroup of $\mathrm{GL}_n(\mathbb{C})$ (or $\mathrm{GL}_n(\mathbb{R})$) for some n.

Remark. Any complex semisimple Lie group can be realized as a linear one, but this is not true for real Lie groups (e.g. the universal cover of $\mathrm{SL}_2(\mathbb{R})$).

Proposition 13.1. *Suppose $\mathfrak{g}_\theta$ is a real form of a semisimple Lie algebra $\mathfrak{g}$, and let G be the connected complex Lie group corresponding to $\mathfrak{g}$. Define $G_\theta = G^{\omega_\theta}$. Then G_θ and G_θ° are linear Lie groups, and every connected real semisimple Lie group is of the form G_θ° for some $\mathfrak{g}_\theta$.*

13.2 Maximal Tori

Remark. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{g}^c$ its compact form. Let G be the corresponding connected Lie group and G^c its compact part. Let $\mathfrak{h}^c \subseteq \mathfrak{g}^c$ be a maximal commutative Lie subalgebra. Note that $\mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h} \subseteq \mathfrak{g}$, where $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$.

Remark. Recall that we have $\{(\mathfrak{h}, \Pi)\}$ (here $\mathfrak{h}$ is a Cartan subalgebra and Π the set of simple roots) are all conjugate: For any $(\mathfrak{h}, \Pi)$ and $(\mathfrak{h}', \Pi')$, there exists $g \in G$ such that

$$\mathrm{Ad}_g(\mathfrak{h}, \Pi) = (\mathfrak{h}', \Pi').$$

The same happens for $\mathfrak{g}^c$.

Lemma 13.1. *Any two Cartan subalgebras in $\mathfrak{g}^c$ equipped with systems of simple roots are conjugate under some $g \in G^c$.*

Proof. Fix $(\mathfrak{h}^c, \Pi)$ and $((\mathfrak{h}^c)', \Pi')$. Then there exists $g \in G$ such that

$$\mathrm{Ad}_g(\mathfrak{h}^c, \Pi) = ((\mathfrak{h}^c)', \Pi'),$$

and for $\overline{g} = \omega(g)$, we have $\mathrm{Ad}_{\overline{g}}(\mathfrak{h}^c, \Pi) = ((\mathfrak{h}^c)', \Pi')$. Then $\overline{g}^{-1}g$ commutes with $\mathfrak{h}^c$, so it preserves Π. Now $\overline{g}h = g$ for some $h \in H = \exp(\mathfrak{h}^c_{\mathbb{C}})$. Write $g = kp$ for some $k \in G^c$ and $p \in P$ by the polar decomposition. Then $\overline{g} = kp^{-1}$, so $kp^{-1}h = kp$, so $h = p^2$. So $p = h^{1/2}$ commutes with $\mathfrak{h}^c$ and preserves Π. Thus we can take $k \in G^c$ to be the desired element. $\square$

Definition 13.3. Given a Cartan subalgebra $\mathfrak{h}^c \subseteq \mathfrak{g}^c$, the corresponding Lie group $H^c = \exp(\mathfrak{h}^c) \subseteq G^c$ is called a *maximal torus*.

Corollary 13.0.1. *Any two maximal tori in G or G^c equipped with systems of simple roots are conjugate.*

Theorem 13.1. *Every element of a connected compact Lie group K is contained in a maximal torus, and all maximal tori are conjugate.*

Proof. We can assume K is semisimple and $K = G^c$. Let $K' \subseteq K$ be the set of elements contained in some maximal torus. Fix a maximal torus $T \subseteq K$, and define

$$\begin{aligned} f : K \times T &\longrightarrow K \\ k, t &\longmapsto ktk^{-1}. \end{aligned}$$

Clearly $\operatorname{Im} f = K'$. Note that K' is compact, so K' is closed and hence $K \setminus K'$ is open. Note that a generic $x \in \mathfrak{g}^c$ is a regular element, i.e. its centralizer $\mathfrak{z}_x$ has $\dim \mathfrak{z}_x = \operatorname{rank} \mathfrak{g}$. Then every regular element $g \in K$ is contained in the maximal torus $\exp(\mathfrak{z}_x)$, so the elements of $K' \setminus K$ are non-regular. Since the non-regular elements are defined by a set of polynomial equations, $K' \setminus K$ is also closed. Hence $K' \setminus K$ must be empty by connectedness. $\square$

Corollary 13.1.1. *On compact groups, the exponential map $\exp : \mathfrak{g}^c \to G^c$ is surjective.*

13.3 Semisimple and Unipotent Elements

Definition 13.4. Let G be a connected complex reductive Lie group. An element $g \in G$ is *semisimple* (resp. *unipotent*) if it acts in every finite-dimensional representation of G by semsimple (resp. unipotent, i.e. all eigenvalues are 1) operators.

Exercise 13.1. Recall that for any compact G, there is a faithful finite-dimensional representation Y. Show that $g \in G$ is semisimple (resp. unipotent) if and only if it acts on Y by a semisimple (resp. unipotent) operator.

Exercise 13.2. Show that if G is semisimple, then the exponential map defines a homeomorphism between the set of nilpotent elements in $\mathfrak{g} = \mathrm{Lie}(G)$ and the set of unipotent elements in G.

Exercise 13.3. Let Z be the center of a connected complex reductive Lie group G.

1. Show that the homomorphism $\pi : G \to G/Z$ defines a bijection between the unipotent elements of G and G/Z.
2. Show that the set of semisimple elements of G is the preimage under π of the set of semisimple elements in G/Z.

Proposition 13.2 (Jordan decomposition)**.** *Any $g \in G$ has a unique factorization*

$$g = g_s g_u,$$

where $g_s \in G$ is semisimple, $g_u \in G$ is unipotent, and $g_s g_u = g_u g_s$.

Proof. We can reduce to the adjoint group. Then we can decompose

$$\mathrm{Ad}_g = su$$

uniquely on the level of matrices, where s is semisimple and u is unipotent. It then suffices to show that s, u are $\mathrm{Ad}_{g_s}, \mathrm{Ad}_{g_u}$ for some group elements. Fill in the details as an exercise. □

13.4 Cartan Decomposition

Remark. Let G be a connected complex semisimple Lie group. Let $G_\theta \subseteq G$ be a real form, and write $G_\theta = K^c P_\theta$. Then $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$ and $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$ where $\mathfrak{p}_\theta = i\mathfrak{p}^c$.

Proposition 13.3. *We have the following:*

1. *Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}_\theta$. Then the centralizer $\mathfrak{z}$ of $\mathfrak{a}$ in $\mathfrak{g}^c$ has the form $\mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is a reductive Lie algebra in $\mathfrak{k}^c$. Moreover, $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{m}$, $\mathfrak{t} \oplus i\mathfrak{a}$ is a Cartan subalgebra in $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra in $\mathfrak{g}_\theta$.*
2. *If $a \in \mathfrak{a}$ is sufficiently generic in $\mathfrak{p}_\theta$, then the centralizer of a in $\mathfrak{p}_\theta$ is $\mathfrak{a}$.*
3. *For all $p \in \mathfrak{p}_\theta$, there exists $k \in K^c$ such that $\mathrm{Ad}_k(p) \in \mathfrak{a}$.*
4. *All maximal subspaces of $\mathfrak{p}_\theta$ are conjugated by K^c.*

Theorem 13.2 (Cartan decomposition)**.** *Let $\mathfrak{a} \subseteq \mathfrak{p}_\theta$ be a maximal abelian subspace. Let $A = \exp(\mathfrak{a}) \subseteq P_\theta \subseteq G_\theta$, which is isomorphic to $\mathbb{R}^n$ where $n = \dim \mathfrak{a}$. Then $G_\theta = K^c A K^c$.*

Remark. The decomposition in Theorem 13.2 is not unique.

Remark. For $G_\theta = \mathrm{GL}_n(\mathbb{C})$, the Cartan decomposition is $U_1 D U_2$ where U_1, U_2 are unitary and D is diagonal with positive entries, a classical theorem of linear algebra.

Theorem 13.3 (E. Cartan)**.** *Let G_θ be a real form of a connected semisimple complex Lie group. Then any compact subgroup L of G_θ is conjugated to a subgroup of K^c by an element of P_θ. Also, every compact subgroup of G_θ is contained in a maximal one. Therefore, all maximal compact subgroups are conjugate to each other.*

Lecture 14

Mar. 2 — Verma Modules

14.1 Integral Form of Weyl Character Formula

Theorem 14.1. *If f is a conjugation invariant continuous function on a compact connected Lie group K with maximal torus $T \subseteq K$ and Haar probability measures dk and dt, then*

$$\int_K f(k)\,dk = \frac{1}{|W|}\int_T f(t)|\delta(t)|^2\,dt,$$

where $\delta(t) = \rho(t)^{-1}\prod_{\alpha\in R_+}(\alpha(t)-1)$.[1]

Proof. First (exercise) use the Peter-Weyl theorem to show that the characters form a dense set in the space of conjugation invariant functions. So it suffices to show the formula for f a character. Then

$$\int_K \chi_\lambda(k)\,dk = \delta_{\lambda,0}$$

is the left-hand side by the orthogonality of characters. Now write

$$\chi_\lambda(t) = \frac{\sum_{w\in W}\varepsilon(w)t^{w(\lambda+\rho)}}{\Delta}, \quad \Delta = \sum_{w\in W}\varepsilon(w)t^{w(\rho)} = t^{-\rho}\prod_{\alpha\in R_+}(t^\alpha - 1) = \rho(t)^{-1}\prod_{\alpha\in R_+}(\alpha(t)-1),$$

so $\Delta = \delta(t)$. Here $\varepsilon(w) = (-1)^{\ell(w)}$ is the sign of w. Then we get

$$\frac{1}{|W|}\int_T \chi_\lambda(t)\overline{\chi_\mu(t)}|\delta(t)|^2\,dt = \frac{1}{|W|}\int_T \sum_{w\in W}\varepsilon(w)t^{w(\lambda+\rho)}\sum_{w'\in W}\varepsilon(w')t^{-w'(\mu+\rho)}\,dt$$

This is equal to $\delta_{\lambda,\mu}$ by the orthogonality of characters, so we get the formula. $\square$

Remark. If $T \cong \mathbb{R}^r/\Lambda$ where $\Lambda \cong \mathbb{Z}^r$ and $\mathcal{F}$ is a fundamental domain for T, then $H = \sum_{i=1}^r x_i e_i$ and

$$dt = \frac{dx_1\ldots dx_r}{\operatorname{vol}(\mathcal{F})}.$$

For $e^{i\theta_j}$ on $U(n)$, this is given by $(d\theta_1\ldots d\theta_n)/(2\pi)^n$.

Exercise 14.1. Let f be a conjugation invariant function on $U(n)$. Show that

$$\int_{U(n)} f(k)\,dk = \frac{1}{(2\pi)^n n!}\int_{|z_1|=\cdots=|z_n|=1} f(\operatorname{diag}(z_1,\ldots,z_n))\prod_{m<j}|z_m - z_j|^2\,d\theta_1\ldots d\theta_n, \quad z_j = e^{i\theta_j}.$$

[1] Recall that if $\alpha\in R_+$ and $e^H = t$ for $H \in \mathfrak{t} = \operatorname{Lie} T$, then we denote $e^{\alpha(H)} = t^\alpha = \alpha(t)$.

14.2 More on Verma Modules

Remark. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Let $\lambda \in \mathfrak{h}^*$ and M_λ the Verma module of highest weight λ, i.e. it is generated by one vector v_λ with

$$e_i v_\lambda = 0 \quad \text{and} \quad h(v_\lambda) = \lambda(h) v_\lambda$$

for $1 \leq i \leq r$. Note that $M_\lambda = \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional representation for $\mathfrak{h} \oplus \mathfrak{n}^+$ defined by by the above actions. Note that M_λ is irreducible for generic λ. In these cases,

$$L_\lambda = M_\lambda / I_\lambda$$

is irreducible, where I_λ is the maximal proper submodule of M_λ. Note that exchanging e_i and f_i in the above definitions gives the lowest weight Verma modules.

There is a unique symmetric bilinear form on M_λ satisfying the contravariance property:

$$\langle v_\lambda, v_\lambda = 1 \rangle \quad \text{and} \quad \langle e_i u, w \rangle = \langle u, f_i w \rangle$$

for $u, w \in M_\lambda$. This form is known as the *Shapovalov form*, and its kernel is precisely I_λ. Note that if ω is the *Chevalley involution* given by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, and $\omega(h_i) = -h_i$, then

$$\langle xu, v \rangle = -\langle u, \omega(x) v \rangle$$

is the contravariance property from above.

Let $V = \bigoplus V^\mu$ be the weight decomposition (note that the weight spaces are finite-dimensional). Then the V^μ and V^ν are orthogonal with respect to the Shapovalov form for $\mu \neq \nu$. Let

$$V^* = \bigoplus_\lambda (V^\lambda)^*$$

be the *restricted dual* of V, and let $\operatorname{Hom}_{\mathbb{C}}(V_1, V_2)$ be the space of all linear maps.

Proposition 14.1. *If M_λ is irreducible, then its restricted dual is the lowest weight Verma module with lowest weight $-\lambda$.*

Definition 14.1. Define V^c to be the restricted dual of V with action of $\mathfrak{g}$ twisted by ω:

$$\langle xu^c, v \rangle = -\langle u^c, \omega(x) v \rangle, \quad u^c \in V^c,\ v \in V,\ x \in \mathfrak{g}.$$

Naturally, V^c has the same weight subspaces as V. We call M_λ^c the *contragredient Verma module.*

Remark. There is a unique $M_\lambda \to M_\lambda^c$ with image L_λ, and combining it with the pairing above gives

$$M_\lambda^c \otimes M_\lambda \longrightarrow \mathbb{C},$$

which is exactly the Shapovalov form.

Theorem 14.2. *The Verma module is irreducible if and only if the following condition is satisfied: For any $\alpha \in R_+$, we have $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N}$.*

Definition 14.2. We will call λ *generic* if for all $\alpha \in R_+$, we have $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$.

Theorem 14.3. *We have the following:*

1. *The module L_λ is finite-dimensional if and only if $\lambda \in P^+$.*
2. *The representations L_λ for $\lambda \in P^+$ are pairwise non-isomorphic, and any finite-dimensional irrep is isomorphic to one of them.*
3. *If $\lambda = \sum n_i\omega_i$ with $n_i \in \mathbb{Z}_+$, then L_λ is generated by some v_λ and the action of $\mathfrak{g}$ on v_λ with the constraints $f_i^{n_i+1}v_\lambda = 0$ for $i = 1, \ldots, r$. Moreover, we have $L_\lambda^* \cong L_{-w_0(\lambda)}$.*

Proposition 14.2. *The category $C(\mathfrak{g})$ of finite-dimensional representations is semisimple, i.e. every module V is isomorphic to a direct sum of simple modules L_λ for $\lambda \in P^+$. Moreover, the map*

$$\Phi \longmapsto \Phi(v_\lambda)$$

identifies the multiplicity space $\mathrm{Hom}_{\mathfrak{g}}(L_\lambda, V)$ for $\lambda \in P^+$ with the space

$$(V^\lambda)^{\mathfrak{n}^+} = \{v \in V^\lambda : e_i v = 0\}$$

of singular vectors of weight λ.

14.3 Maximal Roots and the Casimir Operator

Remark. Recall that the *maximal root* θ is the highest weight in the adjoint representation.

Proposition 14.3. *The maximal root is uniquely determined by the following conditions: For all $\alpha \in R$, we have $\theta - \alpha \in Q^+$.*

Remark. We normalize the pairing so that $(\theta, \theta) = 2$. Let

- $\theta = \sum_i h_i\alpha_i$,
- $h = 1 + \sum_i h_i$,
- $h_i^\vee = h_i(\alpha_i, \alpha_i)/2$ (these are the coefficients of the expansion of $\widetilde{\theta}^\vee$),
- $h^\vee = 1 + \sum_i h_i^\vee = (\rho, \theta) + 1$ (this is the dual Coxeter number).

For root systems of type A, D, E, we have $h = h^\vee$.

Remark. Recall that $C = \sum x_i x^i$ is the *Casimir operator*, where $\{x^i\}$ is the dual basis for x_i. Then

$$\Omega = \sum (x_i \otimes x^i)$$

does not depend on the choice of basis, and the value of C on any irreducible highest weight module L_λ is given by $(\lambda, \lambda + 2\rho)$ (and for the lowest weight $L_{-\mu}$ by $(\mu, \mu + 2\rho)$). The value of C in the adjoint representation is given by $2h^\vee$.

Exercise 14.2. Show that in a tensor product of two representations, the action of $\mathfrak{g}$ commutes with Ω.

Proposition 14.4. *We have*

$$\Omega|_{V_\lambda \subseteq V_\mu \otimes V_\nu} = \frac{(\lambda, \lambda + 2\rho) - (\mu, \mu + 2\rho) - (\nu, \nu + 2\rho)}{2}.$$

Proof. The proof relies on the identity

$$\Omega = \frac{1}{2}(\Delta(C) - c \otimes 1 - 1 \otimes c),$$

where $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is given by $x \mapsto x \otimes 1 + 1 \otimes x$. Note that $(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$. $\square$

14.4 Affine Lie Algebras

Remark. Consider a manifold M and a 1-form J satisfying $d(*J) = 0$. The 1-form J takes values in a Lie algebra, and we have

$$[J^a(x), J^b(y)] = f^{a,b,c} J^c(x)\delta(x-y),$$

where the $f^{a,b,c}$ are the structure constants of the Lie algebra. These are called *current algebras*, see the book by J. Mickellson for more details (e.g. for maps $S^3 \to \mathfrak{g}$).

We will consider the case where M is 2-dimensional. Then the conservation property becomes

$$\partial_{\overline{z}} J_z + \partial_z J_{\overline{z}} = 0.$$

In some cases, this can be separated, so that $\partial_{\overline{z}} J_z = 0$, i.e. J_z is holomorphic. Then we can write

$$J_z = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}.$$

The J_n in this expansion are called *modes* (one can think of this as some kind of wave expansion). Then the commutation relations become (with some correction term)

$$[J_n^a, J_m^b] = f^{a,b,c} J_{n+m}^c.$$

Remark. One can think of the modes above as loops. The *loop algebra* is

$$L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}].$$

One can also think of them as maps $S^1 \to \mathfrak{g}$. The commutation relations for $L\mathfrak{g}$ are

$$[x \otimes P, y \otimes Q] = [x, y] \otimes PQ,$$

so that $x \otimes t^n = x[n]$ has commutation relations

$$[x[n], y[m]] = [x, y][m+n].$$

This algebra has a unique nontrivial central extension, i.e. a short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{g} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathbb{C}c \longrightarrow 0$$

Here $\mathfrak{g}$ is a simple Lie algebra. Then on $\widehat{g}$, we have $[c, x \otimes P] = 0$ for $P \in \mathbb{C}[t, t^{-1}]$ and

$$[x \otimes P, y \otimes Q] = [x, y] \otimes PQ + c\frac{(x, y)}{2\pi i} \oint_{|t|=1} Q\, dP,$$

which gives the commutation relation

$$[x[n], y[m]] = [x, y][m+n] + n\delta_{n,-m}(x, y)c.$$

Lecture 15

Mar. 4 — Affine Lie Algebras

15.1 Affine Lie Algebras, Continued

Remark. We can extend $\widehat{\mathfrak{g}}$ again by the grading operator $d = t\frac{d}{dt}$ to get $\widetilde{\mathfrak{g}} = \widehat{\mathfrak{g}} \oplus \mathbb{C}d$, where

$$[d, x[n]] = nx[n] \quad \text{and} \quad [d, x \otimes P] = x \otimes t\frac{t}{dt}P.$$

We can also define a nondegenerate form on $\widehat{g}$ by

$$(x \otimes P, y \otimes Q) = \frac{(x, y)}{2\pi i} \oint_{|t|=1} PQ \, \frac{dt}{t},$$

and on $\widetilde{\mathfrak{g}}$ by $(c, d) = 1$ and $(c, c) = (d, d) = (c, x \otimes P) = (d, x \otimes P) = 0$.

Definition 15.1. Define $\widehat{\mathfrak{h}} \subseteq \widehat{\mathfrak{g}}$ by $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c$ and $\widetilde{\mathfrak{h}} \subseteq \widetilde{\mathfrak{g}}$ by $\widetilde{\mathfrak{h}} = \widehat{\mathfrak{h}} \oplus \mathbb{C}d$, these are Cartan subalgebras. Their duals are

$$\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \quad \text{and} \quad \widetilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta,$$

where $\Lambda_0(c) = 1$, $\Lambda_0(d) = \Lambda_0(h) = 0$ for $h \in \mathfrak{h}$, and $\delta(d) = 1$, $\delta(c) = \delta(h) = 0$ for $h \in \mathfrak{h}$.

Remark. We can write

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} \oplus \bigoplus_{\gamma \in \widehat{R}} \widetilde{\mathfrak{g}}^\gamma,$$

where $\widetilde{\mathfrak{g}}^\gamma = \{x \in \widetilde{\mathfrak{g}} : [h, x] = \gamma(h)x \text{ for } h \in \widetilde{\mathfrak{h}}\}$ and $\widehat{R} = \{\alpha + n\delta : \alpha \in R \sqcup \{0\}, n \in \mathbb{Z} \text{ not both } 0\}$. Then

$$\widetilde{\mathfrak{g}}^{\alpha+n\delta} = \mathfrak{g}^\alpha \otimes t^n \quad \text{and} \quad \widetilde{\mathfrak{g}}^{n\delta} = \mathfrak{h} \otimes t^n,$$

with $\alpha \in R$, $n \in \mathbb{Z}$, and $n \in \mathbb{Z} \setminus \{0\}$, respectively. We can choose a polarization

$$\widehat{R} = \widehat{R}_+ \sqcup \widehat{R}_-, \quad \widehat{R}_+ = \{\alpha + n\delta : n > 0 \text{ or } n = 0, \alpha \in R_+\},$$

where as usual $\widehat{R}_- = -\widehat{R}_+$. The simple roots in $\widehat{R}_+$ are the ones which cannot be represented as a nontrivial sum of positive roots. Let $\alpha_1, \dots, \alpha_r$ be the simple roots of $\mathfrak{g}$ and

$$\alpha_0 = \delta - \theta,$$

where θ is the highest root. Then we can define $\widehat{\mathfrak{n}}^\pm = \mathfrak{n}^\pm \oplus (\mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}])$ and write

$$\widetilde{\mathfrak{g}} = \widehat{\mathfrak{n}}^+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^-.$$

The root lattices $\widehat{Q} = \bigoplus_{i=0}^r \mathbb{Z}\alpha_i$ and $\widehat{Q}^+ = \sum_{i=0}^r \mathbb{Z}_{\geq 0}\alpha_i$. We can define the coroot

$$\alpha_0^\vee = \frac{2h_{\alpha_0}}{(\alpha_0, \alpha_0)} = h_{\alpha_0} = c - \theta^\vee.$$

Then we can define the weight lattice by $\widehat{P} = \langle \Lambda \in \widetilde{\mathfrak{h}}^* : \Lambda(\alpha_i^\vee) \in \mathbb{Z} \rangle = P \oplus \mathbb{Z}\Lambda_0 \oplus \mathbb{C}\delta$, and

$$\widehat{P}^+ = \{\Lambda \in \widetilde{\mathfrak{h}}^* : \Lambda(\alpha_i^\vee) \in \mathbb{Z}_+\} = \{\Lambda = \lambda + k\Lambda_0 - \Delta\delta : \lambda \in P^+, k \in \mathbb{Z}_+, \Delta \in \mathbb{C}, \langle \lambda, \theta^\vee \rangle \leq k\}.$$

From this we get a Cartan matrix $A = \{a_{i,j}\}_{i,j=0}^r$.

15.2 Generalized Cartan Matrices

Remark. A *generalized Cartan matrix* is a matrix $A = \{a_{i,j}\}_{i,j=0}^r$ satisfying

1. $a_{i,i} = 2$ for all i,
2. $a_{i,j} \leq 0$ for $i \neq j$,
3. $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

Definition 15.2. Let A be a generalized Cartan matrix. The *Kac-Moody algebra* $\mathfrak{g}(A)$ is the Lie algebra generated by elements $\{e_i, h_i, f_i\}$ together with the standard Serre relations.

Theorem 15.1. *The Lie algebra $\widehat{\mathfrak{g}}$ is the Kac-Moody algebra associated with the Cartan matrix which we constructed above.*

Remark. We have $e_0 \in \mathfrak{g}^{-\theta} \otimes t$, $f_0 \in \mathfrak{g}^{\theta} \otimes t^{-1}$, $(e_0, f_0) = 1$, and $h_0 = [e_0, f_0] = c - \theta^\vee$.

Remark. Let A be a generalized Cartan matrix and assume that it is symmetrizable, i.e. there exist d_i such that $d_i a_{i,j} = d_j a_{j,i}$. Then we can define an inner product

$$(h_i, h_j) := \frac{a_{i,j}}{d_i}$$

on the Cartan subalgebra. Let Z be the kernel of this inner product. Let

$$\mathfrak{g}_e(A) = \mathfrak{g}(A) \oplus Z^*$$

with commutation relations $[z_1, z_2] = [z_1, h] = 0$, $[z, e_j] = z(\alpha_i)e_i$, and $[z, f_j] = -z(\alpha_j)f_j$. Then

$$\langle h_i, h_j \rangle = \frac{\alpha_{i,j}}{d_j}, \quad \langle h, z \rangle = z(h), \quad \langle z_1, z_2 \rangle = 0$$

defines a nondegenerate pairing on $\mathfrak{g}_e(A)$.

15.3 Verma Modules and Weyl Modules

Definition 15.3. Let $\Lambda \in \widetilde{\mathfrak{h}}^*$. The *Verma module* corresponding to Λ is

$$M_\Lambda = \mathrm{Ind}_{\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+}^{\widetilde{\mathfrak{g}}} \mathbb{C}_\Lambda$$

where $\mathbb{C}_\Lambda$ is the 1-dimensional representation of $\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+$. Let v_Λ be the basis vector, with $hv_\Lambda = \Lambda(h)v_\Lambda$.

Remark. We have a weight decomposition given by

$$M_\Lambda = \bigoplus_{m \in \Lambda - \widehat{Q}^+} M_\Lambda^m,$$

where $\Lambda = \lambda + k\Lambda_0 - \Delta\delta$ for some $\lambda \in \mathfrak{h}^*$. From the $\widehat{g}$ perspective, we have

$$\Delta = \Delta(\lambda) = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}$$

(here $h^\vee$ is the dual Coxeter number). We call k the *central charge* (or *level*) of the representation. In this case, we say that a $\widehat{g}$-module V is of level k or V is a module with central charge k. We denote the corresponding Verma module M_Λ by $M_{\lambda,k}$.

The case $k = -h^\vee$ is called the *critical level*, though this generally does not happen.

Remark. The Verma module $M_{\lambda,k}$ has a natural $\mathbb{Z}$-grading given by

$$M_{\lambda,k} = \bigoplus_{n \geq 0} M_{\lambda,k}[-n],$$

where $M_{\lambda,k}[-n]$ is naturally a $\mathfrak{g}$-module and an eigenspace of d with eigenvalue $-n - \Delta(\lambda)$.

Example 15.3.1. We have $M_{\lambda,k}[0] = M_\lambda$, the Verma module for $\mathfrak{g}$.

Remark. Let $L_\Lambda = M_\Lambda / I_\Lambda$, where I_Λ is the maximal proper submodule of M_Λ. Then $L_{\lambda,k} := L_\Lambda$ is an irreducible module. The *Chevalley involution* $\widetilde{\omega}$ is given by

$$\widetilde{\omega}(d) = -d$$

and on $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ by $t \mapsto t^{-1}$. Then $\widetilde{\omega}$ gives a contravariant *Shapovalov form* with kernel I_Λ.

Remark. We can construct a class of representations of $\widetilde{\mathfrak{g}}$ induced from $\mathfrak{g}$-modules. Let

$$\widetilde{\mathfrak{g}}^+ = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d \subseteq \widetilde{\mathfrak{g}}.$$

If V is a module for $\mathfrak{g}$, then automatically V is also a module for $\widetilde{\mathfrak{g}}^+$ by letting

1. $\mathfrak{g} \otimes t\mathbb{C}[t]$ act by 0,
2. c, d act by $k, -\Delta$, respectively.

Then one can define $\operatorname{Ind}_{\widetilde{g}^+}^{\widetilde{\mathfrak{g}}} V$, which is a module for $\widetilde{\mathfrak{g}}$.

Example 15.3.2. Verma modules belong to this class, as we can write

$$M_{\lambda,k} = \operatorname{Ind}_{\widetilde{\mathfrak{g}}^+}^{\widetilde{\mathfrak{g}}} M_\lambda$$

where d acts on M_λ by $-\Delta(\lambda)$.

Example 15.3.3. Let $V_{\lambda,k} = \operatorname{Ind}_{\widetilde{\mathfrak{g}}^+}^{\widetilde{\mathfrak{g}}} L_\lambda$, where d acts as $-\Delta(\lambda)$. If $\lambda \in P_+$, we call $V_{\lambda,k}$ a *Weyl module*.

Remark. We get a contragradient Verma module by replacing ω with $\widetilde{\omega}$. This gives maps $M_{\lambda,k} \to M_{\lambda,k}^c$ and $V_{\lambda,k} \to V_{\lambda,k}^c$, which are isomorphisms if $M_{\lambda,k}, V_{\lambda,k}$ are irreducible.

15.4 Integrable Representations

Theorem 15.2. *The Verma module M_Λ is irreducible if the following condition is satisfied: For every $\widehat{\alpha} \in \widehat{R}_+$ and $N \in \{1, 2, \dots\}$, we have $(\Lambda + \widehat{\rho}, \widehat{\alpha}) \neq N(\widehat{\alpha}, \widehat{\alpha})/2$, where $\widehat{\rho} = \rho + h^\vee \Lambda_0$.*

Remark. For $(\widehat{\alpha}, \widehat{\alpha}) \neq 0$, the above condition becomes $(\Lambda + \widehat{\rho}, \widehat{\alpha}^\vee) \neq N$, which is the analogue of the simple Lie algebra case.

Remark. We call weights satisfying $(\Lambda + \widehat{\rho}, \widehat{\alpha}) \neq N(\widehat{\alpha}, \widehat{\alpha})/2$ *generic*. For generic weights, $L_\Lambda = M_\Lambda$.

Theorem 15.3. *Let $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$ be such that $k \notin \mathbb{Q}\langle \alpha^\vee, \lambda \rangle + \mathbb{Q}$ for all $\alpha \in R_+$. Then the induced module $V_{\lambda,k}$ is irreducible. If this holds, we say that k is* generic with respect to λ.

Corollary 15.3.1. *If $\lambda \in P^+$ and $k \notin \mathbb{Q}$, then the Weyl module $V_{\lambda,k}$ is irreducible.*

Lecture 16

Mar. 9 — Affine Lie Algebras, Part 2

16.1 Integrable Representations, Continued

Remark. Recall that the Weyl module is given by

$$V_{\lambda,k} = \operatorname{Ind}_{\widetilde{\mathfrak{g}}^+}^{\widetilde{\mathfrak{g}}} L_\lambda,$$

where L_λ is an irreducible representation for $\mathfrak{g}$ and $\lambda \in P^+$. When $k \notin \mathbb{Q}\langle \alpha^\vee, \lambda\rangle + \mathbb{Q}$ for any $\alpha \in R_+$, we have that $V_{\lambda,k}$ is irreducible.

Definition 16.1. A highest weight representation V of $\widetilde{\mathfrak{g}}$ is *integrable* if for all $u \in V$ and $0 \leq i \leq r$,

$$f_i^{n_i+1} u = 0$$

for some positive integer n_i.

Remark. The action of $\widetilde{\mathfrak{g}}$ on integrable representations can be extended to the corresponding centrally extended loop group.

Remark. Integrable representations are the analogues of finite-dimensional representations for $\widetilde{\mathfrak{g}}$.

Theorem 16.1. *We have the following:*

1. *The module L_Λ is integrable if and only if $\Lambda \in \widehat{P}^+$.*
2. *The representations L_Λ for $\Lambda \in \widehat{P}^+$ are pairwise nonisomorphic, and any highest weight integrable module is isomorphic to one of the L_Λ.*
3. *If Λ is a dominant integral weight and $n_i = \langle \Lambda, \alpha_i^\vee\rangle$, then the $\widehat{\mathfrak{g}}$-module L_Λ is generated by a highest weight vector v_Λ, with relations (for $0 \leq i \leq r$)*

$$e_i v_\Lambda = 0, \quad h_i v_\Lambda = n_i v_\Lambda, \quad f_i^{n_i+1} v_\Lambda = 0.$$

Remark. Let V be a module for $\mathfrak{g}$. For a fixed $z \in \mathbb{C}^\times$, define

$$(x \otimes P(t))u = P(z)xu$$

for $u \in V$. If V is finite-dimensional, then this representation $V(z)$ is an integrable $\widehat{\mathfrak{g}}$-module (called an *evaluation module*). Note that $V(z) = V[z, z^{-1}]$. Let $d = z(d/dz)$. Then we get a $\widetilde{\mathfrak{g}}$-action on

$$z^{-\Delta} V[z, z^{-1}] = V \otimes z^{-\Delta}\mathbb{C}[z, z^{-1}],$$

where d acts by $-\Delta + n$.

16.2 The Virasoro Algebra

Remark. The *Witt algebra* is generated by L_n (for $n \in \mathbb{Z}$) with relations

$$[L_m, L_n] = (m-n)L_{m+n}.$$

If we write $L_n = -z^{n+1}(d/dz)$ for $z = e^{it}$, then the Witt algebra can be realized as vector fields on a circle. Consider $z \mapsto z + tz^{n+1}$ and $f(z) \mapsto f(z + tz^{n+1})$. The *Virasoro algebra* Vir is generated by the L_n and an additional element K, with relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}K, \quad [K, L_n] = 0.$$

The Witt algebra and the Virasoro algebra act on $\widehat{\mathfrak{g}}$ as follows:

$$[L_m, x[n]] = -nx[m+n] \quad \text{and} \quad [L_n, c] = [K, \cdot] = 0.$$

Note that the action of L_0 coincides with the action of $-d$. Thus we have a semidirect product $\mathrm{Vir} \ltimes \widehat{\mathfrak{g}}$, which contains $\widetilde{\mathfrak{g}}$ as a subalgebra.

Let V be a highest weight representation of $\widehat{\mathfrak{g}}$ at level k, with $k \neq -h^\vee$. Define

$$L_m = \frac{1}{2(k+h^\vee)} \sum_{a\in B}\sum_{n\in\mathbb{Z}} {:}a[n]a[m-n]{:} \quad \text{and} \quad K = \frac{k \dim \mathfrak{g}}{k+h^\vee}\,\mathrm{id}, \tag{$*$}$$

where B is an orthonormal basis for $\mathfrak{g}$. Here ${:}a[n]a[k]{:}$ is the *normal ordered product*

$${:}a[n]a[k]{:} = \begin{cases} a[n]a[k] & \text{if } k \geq n, \\ a[k]a[n] & \text{if } k < n, \end{cases}$$

i.e. modes which kill the highest weight vector are pushed all the way to the right.

Theorem 16.2. *The formulas* ($*$) *define an action of the Virasoro algebra on* V*. This action with the original action of* $\widehat{\mathfrak{g}}$ *combine into a* $(\mathrm{Vir} \ltimes \widehat{\mathfrak{g}})$*-action on* V*.*

Remark. Let v be the highest weight vector. Then

$$L_0 v = \frac{1}{2(k+h^\vee)} \sum_{n\in\mathbb{Z}}\sum_{a\in B} {:}a[n]a[-n]{:}v = \frac{1}{2(k+h^\vee)} \sum_{a\in B} {:}a[0]a[0]{:}v = \frac{(\lambda, \lambda+2\rho)}{2(k+h^\vee)}v,$$

where $\sum_{a\in B} {:}a[0]a[0]{:} = \sum_{a\in B} a^2$ is the Casimir operator. Thus we can choose Δ so that $d = -L_0$.

16.3 Currents

Definition 16.2. Let $x[n] = x \otimes t^n$ for $x \in \mathfrak{g}$. The *current* associated to x is

$$J_x(z) = \sum_{n\in\mathbb{Z}} x[n]z^{-n-1} = J_x^+(z) - J_x^-(z),$$

where $J_x^+(z) = \sum_{n<0} x[n]z^{-n-1}$ and $J_x^-(z) = -\sum_{n\geq 0} x[n]z^{-n-1}$.

Remark. Note that $J_x^-(z)$ is a well-defined operator on a highest weight module V. Similarly, the products $J_x^+(z)J_y^+(\zeta)$, $J_x^-(z)J_y^-(\zeta)$, and $J_x^+(z)J_y^-(\zeta)$ are well-defined as operators on $\widehat{V}$. On the other hand, $J_x^-(z)J_y^+(\zeta)$ is not always well-defined: The components of $J_x^-(z)J_y^+(\zeta)u$ are of the form

$$-\sum_{j\geq j_0} z^{-j-1}w^{-n+j-1}x[j]y[n-j]u. \qquad (**)$$

Nevertheless, $x[j]u = 0$ for j sufficiently large, and so

$$x[j]y[n-j]u = [x[j], y[n-j]]u = ([x,y])[n]u + j\langle x,y\rangle\delta_{n,0}ku.$$

Thus, analytically, the convergence of the series $(**)$ amounts to the convergence of

$$\sum_{j\geq j_0} z^{-j-1}\zeta^{-n+j-1} \quad \text{and} \quad \sum_{j\geq j_0} jz^{-j-1}\zeta^{-n+j-1}.$$

These converge for $|z| > |\zeta|$, so the operator $J_x^-(z)J_y^+(\zeta)$ is well-defined on $\widehat{V}$ in this region.

Theorem 16.3. *We have the following:*

$$[J_x^\pm(z), J_y^\pm(w)] = \frac{1}{z-w}(J_{[x,y]}^\pm(z) - J_{[x,y]}^\pm(w)), \quad z \neq w;$$

$$[J_x^-(z), J_y^+(w)] = \frac{1}{z-w}(J_{[x,y]}^-(z) - J_{[x,y]}^+(w)) - \frac{k\langle x,y\rangle}{(z-w)^2}, \quad |z| > |w|.$$

Corollary 16.3.1. *We have the following:*

$$[J_x^\pm(z), J_y^\pm(z)] = \frac{d}{dz}J_{[x,y]}^\pm(z).$$

Remark. Let $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. Then the formulas $(*)$ from Section 16.2 can be rewritten as

$$L(z) = \frac{1}{2(k+h^\vee)}\sum_{a\in B} {:}J_a(z)^2{:},$$

where the normal ordered product is extended $\mathbb{C}[z,z^{-1}]$-linearly.

Lecture 17

Mar. 11 — KZ Equations

17.1 Classification of Intertwining Operators

Remark. Let $L_{\lambda_1}, L_{\lambda_0}$ be irreps of $\mathfrak{g}$ and k a complex number generic with respect to λ_1, λ_0. Then the induced modules $V_{\lambda_i,k} = \mathrm{Ind}_{\widetilde{g}^+}^{\widetilde{g}} L_{\lambda_i}$ are irreducible, and $V_{\lambda_i,k} = L_{\lambda_i,k}$. Note that $V_{\lambda_i,k}[0] = L_{\lambda_i}$.

We will be interested in $\widehat{\mathfrak{g}}$-intertwining operators $\Phi : V_{\lambda_1,k} \to V_{\lambda_0,k} \widehat{\otimes} V(z)$, where $V(z)$ is an evaluation module and $\widehat{\otimes}$ denotes completed tensor product, i.e. elements of $V_{\lambda_0,k} \widehat{\otimes} V(z)$ take the form $\sum_{i=1}^{\infty} w_i \otimes v_i$, where $w_i \in V_{\lambda_0,k}$ are homogeneous with $\deg(w_i) \to -\infty$ and $v_i \in V(z)$. Note that we must have

$$\Phi x[n] = ((x[n] \otimes 1) + z^n(1 \otimes x))\Phi$$

by the assumption that Φ is an intertwining operator.

Theorem 17.1. *Let $g : L_{\lambda_1} \to L_{\lambda_0} \otimes V$ be a $\mathfrak{g}$-homomorphism. For generic k, there exists a unique $\widehat{\mathfrak{g}}$-intertwining operator*

$$\Phi^g(z) : V_{\lambda_1,k} \longrightarrow V_{\lambda_0,k} \widehat{\otimes} V(z)$$

such that for all $w \in V_{\lambda_1,k}[0]$, the degree 0 component of $\Phi^g(z)w$ is equal to gw.

Proof. For simplicity assume that V is finite-dimensional, the proof for infinite-dimensional V is similar. We construct an operator $\Phi^g(z)$ satisfying the given conditions uniquely. Let $w \in V_{\lambda_1,k}[0] = L_{\lambda_1}$. Then $\Phi^g(z)w$ is annihilated by $\mathfrak{g} \otimes t\mathbb{C}[t]$ and the degree 0 component of $\Phi^g(z)w$ is gw. Thus

$$\Phi^g(z)w \in (V_{\lambda_0,k} \widehat{\otimes} V(z))^{\mathfrak{g}\otimes t\mathbb{C}[t]} \cong \mathrm{Hom}_{\mathfrak{g}\otimes t\mathbb{C}[t]}(V_{\lambda_0,k}^*, V(z)).$$

Note that $V_{\lambda_0,k}^*$ is freely generated by $(V_{\lambda,k}[0])^* = L_{\lambda_0}^*$ over $\mathfrak{g} \otimes t\mathbb{C}[t]$ for generic k, so $\Phi^g(z)w$ is uniquely determined by its restriction in $\mathrm{Hom}_{\mathbb{C}}(L_{\lambda_0}^*, V) \cong L_{\lambda_0} \otimes V$, i.e. its degree 0 component. Using

$$\Phi^g(z)x[n]w = ((x[n] \otimes 1) + z^n(1 \otimes x))\Phi^g(z)w$$

for $w \in V_{\lambda_1,k}$ and $n < 0$, we can extend $\Phi^g(z)$ to all of $V_{\lambda_1,k}$ inductively, as $V_{\lambda_1,k}$ is generated by its degree 0 component over $\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$. This defines the operator $\Phi^g(z)$. Now note that no choices were allowed in the above construction, hence $\Phi^g(z)$ must be unique. $\square$

Proposition 17.1. *For any $\Delta \in \mathbb{C}$, the expression $z^{-\Delta}\Phi^g(z)$ is also a $\widehat{\mathfrak{g}}$-intertwiner*

$$V_{\lambda_1,k} \longrightarrow V_{\lambda_0,k} \widehat{\otimes} z^{-\Delta} V[z, z^{-1}].$$

It is a $\widetilde{\mathfrak{g}}$-homomorphism if and only if $\Delta = \Delta(\lambda_1) - \Delta(\lambda_0)$.

Proof. Note that $z^{-\Delta}\Phi^g(z)$ is a $\widetilde{\mathfrak{g}}$-homomorphism if and only if

$$z^{-\Delta}\Phi^g(z)d - (d\otimes 1)z^{-\Delta}\Phi^g(z) = \left(1\otimes z\frac{d}{dz}\right)z^{-\Delta}\Phi^g(z).$$

Since $z^{-\Delta}\Phi^g(z) = \sum_{n\in\mathbb{Z}}\Phi^g[n]z^{-n-\Delta}$ and $\Phi^g[n]$ has degree n, it suffices to have the above identity in degree 0, which is given by $-\Delta(\lambda_1)+\Delta(\lambda_0) = -\Delta$. □

Remark. Denote $\widetilde{\Phi}^g(z) = z^{-\Delta}\Phi^g(z) = \sum_{n\in\mathbb{Z}}\Phi^g[n]z^{-n-\Delta}$ for $\Delta = \Delta(\lambda_1)-\Delta(\lambda_0)$.

17.2 Operator KZ Equations

Remark. Let V^* be the restricted dual of V, so that $V^*(z)\cong(V(z))^*$ as $\widetilde{\mathfrak{g}}$-modules. Let $u\in V^*$, and define $\Phi^g_u(z)w = u(\Phi^g(z)w)$ for $w\in V_{\lambda_1,k}$, so that $\Phi^g_u(z)$ is an operator $V_{\lambda_1,k}\to\widehat{V}_{\lambda_0,k}$. The intertwining relation is given by $[x[n],\Phi^g_u(z)] = z^n\Phi^g_{xu}(z)$. Note that the currents act on $V^*(z)$ by

$$J_x^+(\zeta)u = \sum_{n>0}\zeta^{n-1}x[-n]u = \sum_{n>0}\zeta^{n-1}z^{-n}xu = \frac{xu}{z-\zeta},\quad |\zeta|<|z|,$$
$$J_x^-(\zeta)u = -\sum_{n\le 0}\zeta^{n-1}x[-n]u = -\sum_{n\le 0}\zeta^{n-1}z^{-n}xu = \frac{xu}{z-\zeta},\quad |z|<|\zeta|,$$

and in terms of currents, the intertwining relation is given by

$$[J_x^\pm(\zeta),\Phi^g_u(z)] = \frac{1}{z-\zeta}\Phi^g_{xu}(z). \tag{$*$}$$

Let $V=V_\mu$ be a lowest weight $\mathfrak{g}$-module with lowest $-\mu$, so that the Casimir element C acts on V_μ by the scalar $(\mu,\mu+2\rho)$. Let $\widehat{\Phi}^g(z) = z^{-\Delta(\mu)}\widetilde{\Phi}^g(z) = \sum_{n\in\mathbb{Z}}\Phi^g[n]z^{-n-\Delta}$ for $\Delta = \Delta(\lambda_1)-\Delta(\lambda_0)+\Delta(\mu)$.

For $a\in\mathfrak{g}$, define the *normal ordered product*

$$:J_a(z)\widehat{\Phi}^g_u(z): = J_a^+(z)\widehat{\Phi}^g_u(z) - \widehat{\Phi}^g_u(z)J_a^-(z).$$

The following result is known as the *operator Knizhnik-Zamolodchikov (KZ) equation.*

Theorem 17.2 (Frenkel-Reshetikhin)**.** *Let B be an orthonormal basis for $\mathfrak{g}$. Then*

$$(k+h^\vee)\frac{d}{dz}\widehat{\Phi}^g_u(z) = \sum_{a\in B}:J_a(z)\widehat{\Phi}^g_{au}(z):.$$

Proof. Since $\widetilde{\Phi}^g(z)$ is invariant under d, we have $z(d/dz)\widetilde{\Phi}^g_u(z) = -[d,\widetilde{\Phi}^g_u(z)]$, or equivalently,

$$z\frac{d}{dz}\widehat{\Phi}^g_u(z) = -[d,\widehat{\Phi}^g_u(z)] - \Delta(\mu)\widehat{\Phi}^g_u(z).$$

Using the expansion for $d=-L_0$, the above equality becomes

$$z\frac{d}{dz}\widehat{\Phi}^g_u(z) = \frac{1}{2(k+h^\vee)}\sum_{a\in B}\Bigg(\underbrace{\sum_{n\le 0}[a[n]a[-n],\widehat{\Phi}^g_u(z)] + \sum_{n>0}[a[-n]a[n],\widehat{\Phi}^g_u(z)]}_{S(a)}\Bigg) - \Delta(\mu)\widehat{\Phi}^g_u(z).$$

By the intertwining relation $(*)$, we can rewrite the inner sum above as

$$\begin{aligned}S(a) &= \sum_{n\le 0}(z^{-n}a[n]\widehat{\Phi}^g_{au}(z) + z^n\widehat{\Phi}^g_{au}(z)a[-n]) + \sum_{n>0}(z^n a[-n]\widehat{\Phi}^g_{au}(z) + z^{-n}\widehat{\Phi}^g_{au}(z)a[n]) \\ &= 2zJ_a^+(z)\widehat{\Phi}^g_{au}(z) - 2z\widehat{\Phi}^g_{au}(z)J_a^-(z) + a[0]\widehat{\Phi}^g_{au}(z) - \widehat{\Phi}^g_{au}(z)a[0] = 2z{:}J_a(z)\widehat{\Phi}^g_{au}(z){:} + \widehat{\Phi}^g_{a^2u}(z).\end{aligned}$$

Substituting this back in, we get

$$z\frac{d}{dz}\widehat{\Phi}^g_u(z) = \frac{1}{k+h^\vee}\sum_{a\in B}\left(z{:}J_a(z)\widehat{\Phi}^g_{au}(z){:} + \frac{1}{2}\widehat{\Phi}^g_{a^2u}(z)\right) - \Delta(\mu)\widehat{\Phi}^g_u(z).$$

Now $\sum_{a\in B}\widehat{\Phi}^g_{a^2u}(z) = \widehat{\Phi}^g_{Cu}(z) = (\mu,\mu+2\rho)\widehat{\Phi}^g_u(z)$, so the last two terms cancel, and we have

$$z\frac{d}{dz}\widehat{\Phi}^g_u(z) = \frac{1}{k+h^\vee}\sum_{a\in B} z{:}J_a(z)\widehat{\Phi}^g_{au}(z){:}.$$

Rearranging gives the desired equation from here. □

Proposition 17.2. *We have the following:*

$$[L_m, \widehat{\Phi}^g_u(z)] = \frac{z^m}{k+h^\vee}{:}J(z)\widehat{\Phi}^g_u(z){:} + mz^m\Delta(\mu)\widehat{\Phi}^g_u(z).$$

Lecture 18

Mar. 16 — KZ Equations, Part 2

18.1 KZ Equations for Correlation Functions

Remark. Recall that if we write $J_x(z) = J_x^+(z) - J_x^-(z)$, then

$$[J_x^{\pm}(z), \Phi_n^{(g)}(w)] = \frac{1}{z-w}\Phi_{xn}^g(w).$$

Then $\widehat{\Phi}^g(z) = z^{-\Delta(\mu)}\widetilde{\Phi}^g(z) = \sum_{n\in\mathbb{Z}} \Phi^g[n]z^{-n-\Delta}$, where $\Delta = \Delta(\lambda_1) - \Delta(\lambda_0) + \Delta(\mu)$ and

$$\Phi^g(z) : V_{\lambda_1,k} \longrightarrow V_{\lambda_0,k} \widehat{\otimes} V_\mu(z).$$

Theorem 18.1 (FR, also Theorem 17.2)**.** *We have*

$$(k+h^\vee)\frac{d}{dz}\widehat{\Phi}_u^g(z) = \sum_{a\in B} {:}J_a(z)\widehat{\Phi}_{au}^g(z){:}.$$

Remark. Let $\widehat{\Phi}^{g_i}(z_i) : V_{\lambda_i,k} \widehat{\otimes}\, z^{-\Delta}V_{\mu_i}[z_i^{\pm 1}]$ and

$$\psi(z_1,\ldots,z_N) = (\widehat{\Phi}^{g_1}(z_1)\otimes\cdots\otimes 1)\cdots(\widehat{\Phi}^{g_{N-1}}(z_{N-1})\otimes 1)\widehat{\Phi}^{g_N}(z_N),$$

where $\psi : V_{\lambda_N,k} \to V_{\lambda_0,k}\widehat{\otimes} V_{\mu_1}(z_1)\widehat{\otimes}\cdots\widehat{\otimes} V_{\mu_N}(z_N)$. Here V_{μ_i} are the lowest weight modules with lowest weight $-\mu_i$. Let $u_0 \in (V_{\lambda_0,k}[0])^* = L_{\lambda_0}^*$ and $u_{N+1} \in V_{\lambda_N,k}[0] = L_{\lambda_N}$. Then

$$\langle u_0, \psi(z_1,\ldots,z_N)u_{N+1}\rangle \in V_{\mu_1}\otimes\cdots\otimes V_{\mu_N}.$$

We can also treat this as lying in $z_1^{-\Delta_1}\cdots z_N^{-\Delta_N}\mathbb{C}[\![z_2/z_1,\ldots,z_N/z_{N-1}]\!]$. Fix u_0 and define a V-valued function (called the *correlation function*)

$$\psi(z_1,\ldots,z_N) = \langle u_0, \psi(z_1,\ldots,z_N)\cdot\rangle,$$

where $V = V_{\mu_1}\otimes\cdots\otimes V_{\mu_N}\otimes L_{\lambda_N}^*$. For $u_i \in V_{\mu_i}^*$, we have

$$\psi_{u_1,\ldots,u_{N+1}}(z_1,\ldots,z_N) = \langle u_0, \Phi_{u_1}^{g_1}\cdots\Phi_{u_N}^{g_N}u_{N+1}\rangle \in \mathbb{C}.$$

Theorem 18.2. *The correlation function $\psi(z_1,\ldots,z_N)$ satisfies the following system of differential equations:*

$$(k+h^\vee)\partial_{z_i}\psi = \left(\sum_{j=1,j\neq i}^N \frac{\Omega_{i,j}}{z_i-z_j} + \frac{\Omega_{i,N+1}}{z_i}\right)\psi,$$

where $\Omega = \sum x_i\otimes x^i$ and $A_{i,j} = (x)_i(y)_j$ for $A = x\otimes y \in \mathfrak{g}\otimes\mathfrak{g}$.

Proof. We can write

$$\begin{aligned}
&(k+h^\vee)\frac{\partial}{\partial z_i}\psi_{u_1,\dots,u_{N+1}}(z_1,\dots,z_N)\\
&\quad=\sum_{a\in B}\langle u_0,\widehat{\Phi}_{u_1}(z_1),\dots,(J_a^+(z_i)\widehat{\Phi}_{au_i}(z_i)-\widehat{\Phi}_{au_i}(z_i)J_a^-(z_i)),\dots,\widehat{\Phi}_{u_N}(z_N)u_{N+1}\rangle\\
&\quad=\sum_{j=1}^{i-1}\langle u_0,\widehat{\Phi}_{u_1}(z_1),\dots,[\widehat{\Phi}_{u_j}(z_j),J_a^+(z_i)],\dots,\widehat{\Phi}_{u_N}(z_N)u_{N+1}\rangle && (*)\\
&\qquad+\sum_{j=i+1}^{N}\langle u_0,\widehat{\Phi}_{u_1}(z_1),\dots,[\widehat{\Phi}_{u_j}(z_j),J_a^-(z_i)],\dots,\widehat{\Phi}_{u_N}(z_N)u_{N+1}\rangle && (**)\\
&\qquad+\langle u_0,J_a^+(z_i)\widehat{\Phi}_{u_1}(z_1),\dots,\widehat{\Phi}_{au_i}(z_i),\dots,\widehat{\Phi}_{u_N}(z_N)u_{N+1}\rangle && (***)\\
&\qquad-\langle u_0,\widehat{\Phi}_{u_1}(z_1),\dots,\widehat{\Phi}_{au_i}(z_i),\dots,\widehat{\Phi}_{u_N}(z_N)J_a^-(z_i)u_{N+1}\rangle.
\end{aligned}$$

Now we have $J_a^-(z)u_{N+1}=(-1/z)au_{N+1}$ and $\langle u_0,J_a^+(z)w\rangle=0$ since $w\in V_{\lambda,k}$. Then

$$(*)=\sum_{j=1}^{i-1}\sum_{a\in B}\frac{1}{z_i-z_j}\langle u_0,\widehat{\Phi}_{u_1}(z_1),\dots,\widehat{\Phi}_{au_j}(z_j),\dots,\widehat{\Phi}_{au_i}(z_i),\dots,\widehat{\Phi}_{u_N}(z_N)u_{N+1}\rangle,$$

and similarly,

$$(**)=\sum_{j=i+1}^{N}\sum_{a\in B}\frac{1}{z_i-z_j}\langle u_0,\widehat{\Phi}_{u_1}(z_1),\dots,\widehat{\Phi}_{au_i}(z_i),\dots,\widehat{\Phi}_{au_j}(z_j),\dots,\widehat{\Phi}_{u_N}(z_N)u_{N+1}\rangle.$$

We also get

$$(***)=\sum_{a\in B}\frac{1}{z_i}\langle u_0,\widehat{\Phi}_{u_1}(z_1),\dots,\widehat{\Phi}_{au_i}(z_i),\dots,\widehat{\Phi}_{u_N}(z_N)au_{N+1}\rangle.$$

This is the action of the Casimir element Ω. □

Remark. For $j>i$, we can write

$$\frac{1}{z_i-z_j}=z_i^{-1}\sum_{n=0}^{\infty}(z_j/z_i)^n.$$

Remark. The usual form of the KZ equations (call this the *symmetric* form) is written as

$$(k+h^\vee)\frac{\partial}{\partial z_i}\psi=\Bigg(\sum_{j=1,j\neq i}^{N+1}\frac{\Omega_{i,j}}{z_i-z_j}\Bigg)\psi$$

for $i=1,\dots,N+1$. These systems are equivalent: If $\widetilde{\psi}(z_1,\dots,z_{N+1})$ is a solution to the above system, then $\widetilde{\psi}(z_1,\dots,z_N,0)$ is a solution to the system in Theorem 18.2 (call this the *non-symmetric* form). Similarly, if $\psi(z_1,\dots,z_N)$ is a solution to the system in Theorem 18.2, then $\widetilde{\psi}(z_1,\dots,z_{N+1})=\psi(z_1-z_{N+1},\dots,z_N-z_{N+1})$ is a solution to the above system.

Remark. If $\psi(z_1,\dots,z_{N+1})$ is a solution to the KZ equations, then so is $\psi(z_1+c,\dots,z_{N+1}+c)$ for constant c. Also notice that for $\lambda_N=0$, the non-symmetric system turns into the symmetric one.

Remark. If $u_0 \in (V_{\lambda_0,k}[0])^*$ is the lowest weight vector with respect to $\mathfrak{g}$, then

$$\psi(z_1, \dots, z_N) = \langle u_0, \psi(z_1, \dots, z_N) \cdot \rangle$$

(which a priori has values in $V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_N} \otimes L^*_{\lambda_N}$) takes values in $V^{\mathfrak{n}_-}$. If $\lambda_0 = 0$, then this is $\mathfrak{g}$-invariant, and ψ takes values in $V^{\mathfrak{g}}$.

Theorem 18.3. *The KZ system is consistent, i.e. for all $p, q = 1, \dots, N+1$,*

$$\left[(k + h^\vee)\partial_{z_p} - \sum_{j=1, j\neq p}^{N+1} \frac{\Omega_{p,j}}{z_p - z_j}, (k + h^\vee)\partial_{z_q} - \sum_{j=1, j\neq q}^{N+1} \frac{\Omega_{q,j}}{z_q - z_j}\right] = 0.$$

Proof. We trivially have $\Omega_{p,q} = \Omega_{q,p}$, so

$$[\Omega_{p,q}, \Omega_{p,r} + \Omega_{q,r}] = \left[\sum_a a_p \otimes a_q, \sum_b b_p \otimes b_r + \sum_b b_q \otimes b_r\right] = 0.$$

Expanding this out, we get that

$$0 = \sum_{a,b\in B} (a_q[a_p, b_p]b_r + a_p[a_q, b_q]b_r) = \sum_{b\in B} [\Omega_{p,q}, b_p + b_q]b_r,$$

so that $[\Omega, b \otimes 1 + 1 \otimes b] = 0$. Since $\Omega = \sum_a a \otimes a$, this proves consistency. □

Remark. The above results imply that the KZ equations define a flat connection on a trivial vector bundle V over

$$X_{N+1} = \{(z_1, \dots, z_{N+1}) \in \mathbb{C}^N : z_i \neq z_j\}.$$

We will see later that for any open disk $D \subseteq X_{N+1}$, the space of solutions $\Gamma(D, V_{\mathrm{KZ}})$ is a space of flat sections of V_{KZ} over D. If D is simply connected, then $\Gamma(D, V_{\mathrm{KZ}}) \cong V$ by $\psi \mapsto \psi(p)$. In other words, V_{KZ} is a direct sum of finite dimensional vector bundles (i.e. *local systems*).

Theorem 18.4. *The KZ system is $\mathfrak{g}$-invariant, meaning that if $x \in \mathfrak{g}$ and ψ is a solution, then $x\psi$ is also a solution.*

Proof. Check as an exercise that $[\sum_{j=1,j\neq p}^{N+1} \Omega_{p,j}/(z_p - z_j), x] = 0$, where $x = \sum_{i=1}^{N+1} x_i$. □

18.2 Analyticity of Correlation Functions

Remark. We change variables from $z_1, \dots, z_N$ to $\zeta_i = z_{i+1}/z_i$ for $i = 1, \dots, N-1$, and $\zeta_N = z_N$. Then the KZ equations become the following system:

$$\zeta_i \frac{\partial}{\partial \zeta_i} \psi = A_i(\zeta_1, \dots, \zeta_{N-1})\psi.$$

To see this, one can take logarithmic derivatives and then take differences. Here the A_i are holomorphic functions in the region

$$\{|\zeta_j| < 1 : 1 \leq j \leq N\},$$

and we have $[A_i(0), A_j(0)] = 0$ for all i, j.

Proposition 18.1. *Let $A_i(\zeta_1, \ldots, \zeta_m)$ for $1 \leq i \leq m$ be $n \times n$ matrix-valued functions holomorphic in the region $R = \{|\zeta_j| < 1 : 1 \leq j \leq m\}$ and such that the differential operators $\zeta_i \partial_{z_i} - A_i$ commute with each other. Assume also that for all i, the eigenvalues of $A_i(0)$ do not differ by a nonzero integer. Then the differential equation*

$$\zeta_i \frac{\partial}{\partial \zeta_i} F = A_i(\zeta_1, \ldots, \zeta_m) F, \quad i = 1, \ldots, m$$

has a unique $n \times n$ matrix solution of the form

$$F(\zeta_1, \ldots, \zeta_m) = F_0(\zeta_1, \ldots, \zeta_m) \zeta_1^{A_1(0)} \cdots \zeta_m^{A_m(0)},$$

where F_0 is an $n \times n$ matrix valued function holomorphic in the region R and $F_0(0) = \mathrm{id}$.

Corollary 18.4.1. *In the notation of Proposition 18.1, let f be a vector-valued power series*

$$f(\zeta_1, \ldots, \zeta_m) = \zeta_1^{\delta_1} \cdots \zeta_m^{\delta_m} f_0(\zeta_1, \ldots, \zeta_m),$$

where $f_0 \in \mathbb{C}^n[\![\zeta_1, \ldots, \zeta_m]\!]$ and $\delta_i \in \mathbb{C}$. Then there exists such a vector $v \in \mathbb{C}^n$ such that $f(\zeta_1, \ldots, \zeta_m) = F(\zeta_1, \ldots, \zeta_m)v$, where F is the fundamental solution from Proposition 18.1, and f converges to an analytic solution in the region R.

Proof. Let $v(\zeta_1, \ldots, \zeta_m) = F^{-1} f$, then differentiating gives $\partial_{\zeta_i} v = 0$. □

Theorem 18.5. *The correlation function is absolutely convergent in the region $|z_1| > |z_2| > \cdots > |z_N| > 0$ and defines a regular, multi-valued function in this region.*

Corollary 18.5.1. *For $|z_1| > \cdots > |z_N| > 0$, the product $\widehat{\Phi}_{u_1}(z_1) \cdots \widehat{\Phi}_{u_N}(z_N)$ is a well-defined operator*

$$V_{\lambda_N, k} \longrightarrow \widehat{V}_{\lambda_0, k}$$

(i.e. all matrix elements are converging series).

Corollary 18.5.2. *In the limit $|z_1| \gg \cdots \gg |z_N|$, the correlation function has the following asymptotics:*

$$\psi(z_1, \ldots, z_N) \sim z_1^{-\Delta_1} \cdots z_N^{-\Delta_N} v,$$

where $v = \langle u_0, g_1 \cdots g_N \cdot \rangle \in V$.

Remark. By Corollary 18.5.2, we can write $\psi = z_1^{-\Delta_1} \cdots z_N^{-\Delta_N} \psi_0$, where

$$\psi_0 = \psi_0(\zeta_1, \ldots, \zeta_{N-1}) = \psi_0(z_2/z_1, \ldots, z_N/z_{N-1})$$

is analytic in a neighborhood of $\{\zeta_i = 0\}$ and $\psi_0(0) = v$.

Lecture 19

Mar. 18 — KZ Equations, Part 3

19.1 Correlation Functions, Continued

Proposition 19.1. *The correlation functions span the space of solutions of the KZ equations.*

Remark. Let $\xi = (\lambda_0, \ldots, \lambda_N; g_1, \ldots, g_N; u_0)$, $\lambda_i \in \mathfrak{h}^*$, $g_i \in \operatorname{Hom}_{\mathfrak{g}}(L_{\lambda_i}, L_{\lambda_{i-1}} \otimes V_{\mu_i})$ where $u_0 \in L^*_{\lambda_0}$. Let

$$\Xi = \bigoplus_{\lambda_0, \ldots, \lambda_{N-1}} \left(L^*_{\lambda_0} \otimes \bigotimes_{i=1}^{N} \operatorname{Hom}_{\mathfrak{g}}(L_{\lambda_i}, L_{\lambda_{i-1}} \otimes V_{\mu_i}) \right).$$

The only nonzero terms satisfy $\lambda_i + \mu_i - \lambda_{i-1} \in Q$, and this space has a natural $\mathfrak{g}$-module structure, and the $\operatorname{Hom}_{\mathfrak{g}}(L_{\lambda_i}, L_{\lambda_{i-1}} \otimes V_{\mu_i})$ are trivial $\mathfrak{g}$-modules. Each $\xi \in \Xi$ defines a unique correlation function

$$\psi^{\xi}(z_1, \ldots, z_N) \in V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_N} \otimes L^*_{\lambda_N}.$$

Theorem 19.1. *Fix λ_N and μ_i for $i = 1, \ldots, N$. Let V_{μ_i} be the lowest weight modules satisfying either of the two following conditions:*

1. Generic case: *The weights $\lambda_N, \lambda_N + \mu_N, \ldots, \lambda_N + \mu_N + \cdots + \mu_1$ are generic and k is generic with respect to each;*
2. Finite-dimensional case: *$\lambda_N, \mu_i \in P^+$, $k \notin \mathbb{Q}$, and the V_{μ_i} are irreducible finite-dimensional modules with lowest weight $-\mu_i$.*

Then the correlation functions span the space of V-valued solutions of the KZ equations in a region D. Moreover, the following map

$$\begin{aligned} \Xi &\longrightarrow \Gamma(D, V_{\mathrm{KZ}}) \\ \xi &\longmapsto \psi^{\xi}(z_1, \ldots, z_N) \end{aligned}$$

is an isomorphism of $\mathfrak{g}$-modules.

Remark. In Case 1 of Theorem 19.1, the λ_i are generic, and in Case 2, the $\lambda_i \in P^+$.

Lemma 19.1. *Under assumptions (1) or (2) of Theorem 19.1, the map*

$$(\lambda_0, \ldots, \lambda_{N-1}; g_1, \ldots, g_N; u_0) \longmapsto \langle u_0, g_1 \cdots g_N \cdot \rangle$$

is an isomorphism of $\mathfrak{g}$-modules.

Proof. Note that $\mathrm{Hom}_{\mathfrak{g}}(L_{\lambda_i}, L_{\lambda_{i-1}} \otimes V_{\mu_i}) \cong \mathrm{Hom}_{\mathfrak{g}}(L^*_{\lambda_{i-1}}, V_{\mu_i} \otimes L^*_{\lambda_i})$. So it suffices to show that

$$\bigoplus_v \mathrm{Hom}_{\mathfrak{g}}(L^*_v, V_\mu \otimes L^*_\lambda) \otimes L^*_v \longrightarrow V_\mu \otimes L^*_\lambda$$

is an isomorphism. This is true if we are in the finite-dimensional case. In general, if λ and $\lambda + \mu$ are generic, the L^*_v being irreducible implies that the above map is injective. Since v is generic, $L^*_v \cong M^*_v$ and $L^*_\lambda \cong M^*_\lambda$, so $\mathrm{Hom}_{\mathfrak{g}}(L^*_v, V_\mu \otimes L^*_\lambda) \cong V_\mu^{(\lambda - v)}$ (the weight subspace). □

Corollary 19.1.1. *If u_0 is the lowest weight vector, then the correlation functions span solutions with values in $V^{\mathfrak{n}_-}$. Moreover, if u_0 is $\mathfrak{g}$-invariant, then the correlation functions take values in $V^{\mathfrak{g}}$.*

Remark. We had an identification $\Gamma(D, V_{\mathrm{KZ}}) \cong V$ via $\psi \mapsto \psi(p)$ (i.e. evaluation at p). Now we have

$$\Gamma(D, V_{\mathrm{KZ}}) \cong \Xi \cong V,$$

which is a more sophisticated identification that is not attached to a particular point.

19.2 Trigonometric KZ Equations

Remark. Let $\phi(z_1, \ldots, z_N) = \langle u_0, \psi(z_1, \ldots, z_N) u_{N+1} \rangle \in V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}$. Then

$$\phi \in (V_{\mu_1} \otimes \cdots \otimes V_{\mu_N})^{\lambda_N - \lambda_0},$$

and ϕ satisfies the following differential equation

$$(k + h^\vee) \frac{\partial}{\partial z_i} \phi = \left(\sum_{j=1}^N \frac{\Omega_{ij}}{z_i - z_j} \right) \phi - \frac{1}{z_i} \sum_e (a_e)_i \langle u_0, \psi(z_1, \ldots, z_N) a^\ell u_{N+1} \rangle,$$

where a^e, a_e are dual basis elements in $\mathfrak{g}$. For $\alpha \in R_+$, let $e_\alpha \in \mathfrak{g}^\alpha$, $f_\alpha \in \mathfrak{g}^{-\alpha}$ such that $\langle e_\alpha, f_\alpha \rangle = 1$. Then

$$\Omega = \sum_{\alpha \in R_+} (e_\alpha \otimes f_\alpha + f_\alpha \otimes e_\alpha) + \sum_p x_p \otimes x_p,$$

where x_p is the dual basis in $\mathfrak{h}$, and $C = \sum_{\alpha \in R_+} (e_\alpha f_\alpha + f_\alpha e_\alpha) + \sum_p x_p^2$. We can rewrite

$$\begin{aligned}\sum_e (a_e)_i \langle u_0, \psi a^\ell u_{N+1} \rangle &= \sum_{\alpha \in R_+} (e_\alpha)_i \langle u_0, \psi f_\alpha u_{N+1} \rangle + \sum_p (x_p)_i \rangle u_0, \psi \langle x_p, \lambda_N \rangle u_{N+1} \rangle \\ &= \sum_{\alpha \in R_+} (e_\alpha)_i \sum_{j=1}^N (f_\alpha)_j \phi + (h_{\lambda_N})_i \phi.\end{aligned}$$

Note that u_0 is $\mathfrak{n}_-$-invariant. Now ϕ has weight $\lambda_N - \lambda_0$, so

$$\sum_p \sum_{j=1}^N (x_p)_i (x_p)_j \phi = (h_{\lambda_N - \lambda_0})_i \phi.$$

Then we can rewrite

$$\sum_e (a_e)_i \langle u_0, \psi a^\ell u_{N+1}\rangle = \sum_{j=1}^N \left(\sum_{\alpha\in R_+} (e_\alpha)_i (f_\alpha)_j + \frac{1}{2}\sum_p (x_p)_i (x_p)_j \right)\phi + \frac{1}{2}(h_{\lambda_N} + \lambda_0)_i \phi$$
$$= \sum_{j=1}^N \Omega^+_{i,j}\phi + \frac{1}{2}(h_{\lambda_N+\lambda_0})_i\phi + \left(\sum_{\alpha\in R_+} e_\alpha f_\alpha + \frac{1}{2}\sum_p x_p^2 \right)_i \phi,$$

where $\Omega^+ = \sum_{\alpha\in R_+} e_\alpha \otimes f_\alpha + \frac{1}{2}\sum_p x_p \otimes x_p$

$$\Omega^- = \Omega - \Omega^+ = \sum_{\alpha\in R_+} f_\alpha \otimes e_\alpha + \frac{1}{2}\sum_p x_p \otimes x_p.$$

Check as an exercise that $[e_\alpha, f_\alpha] = h_\alpha$ and that

$$\sum e_\alpha f_\alpha + \frac{1}{2}\sum_p x_p^2 = \frac{1}{2}C + h_\rho.$$

Then we have that

$$\sum_\ell (a_\ell)_i \langle u_0, \psi a^\ell u_{N+1}\rangle = \sum_{\substack{j=1\\ j\neq i}}^N \Omega^+_{i,j}\phi + \frac{1}{2}(h_{\lambda_N+\lambda_0+2\rho})_i\phi + \frac{1}{2}(C)_i\phi$$
$$= \left(\sum_{j=1}^N \Omega^+_{i,j} + \frac{1}{2}(h_{\lambda_N+\lambda_0+2\rho})_i + \frac{1}{2}\langle \mu_i, \mu_i + 2\rho\rangle \right)\phi.$$

Thus we can rewrite the KZ equations as

$$(k+h^\vee) z_i \partial_{z_i}\phi = \left(\sum_{\substack{j=1\\ j\neq i}}^N \frac{z_i\Omega^-_{i,j} + z_j\Omega^+_{i,j}}{z_i - z_j} - \frac{1}{2}(h_{\lambda_N+\lambda_0+2\rho})_i - \frac{1}{2}\langle \mu_i, \mu_i+2\rho\rangle \right)\phi$$

Theorem 19.2. *Let $z_i = e^{z_i}$. Then*

$$(k+h^\vee)\partial_{x_i}\phi = \left(\sum_{\substack{j=1\\ j\neq i}}^N r_{i,j}(x_i - x_j) - \frac{1}{2}(h_{\lambda_N+\lambda_0+2\rho})_i - \langle \mu_i, \mu_i + 2\rho\rangle \right)\phi,$$

where $r(x) = (e^x\Omega^- + \Omega^+)/(e^x - 1)$.

19.3 Factorizable Systems

Remark. The equations we are studying take the form

$$\frac{\partial}{\partial z_i}\phi(z_1,\ldots,z_N) = A_i(z_1,\ldots,z_N)\phi(z_1,\ldots,z_N), \quad \phi \in V,\ A_i \in \mathrm{End}(V),$$

where $\partial_i A_j \partial_j A_i + [A_i, A_j] = 0$ and the A_i are holomorphic in $D \subseteq \mathbb{C}^N$.

Definition 19.1. We say that such a system is *factorizable* if

1. $V = V_1 \otimes \cdots \otimes V_N$, where the $\{V_i\}$ are $\mathfrak{g}$-modules,
2. there exists a function $r(z) \in \mathfrak{g} \otimes \mathfrak{g}$ of one complex variable meromorphic in a neighborhood of the origin, and $\kappa \in \mathbb{C}$ such that

$$A_i(z_1, \ldots, z_N) = \frac{1}{\kappa}\left(\sum_{j>i} r_{i,j}(z_i - z_j) - \sum_{j<i} r_{j,i}(z_j - z_i) + s_i\right)$$

for $s \in \mathfrak{g}$ such that $[s \otimes 1 + 1 \otimes s, r(z)] = 0$.

Proposition 19.2. *A factorizable system is consistent if*

$$[r_{i,j}(z_i - z_j), r_{j,k}(z_j - z_k)] + [r_{i,k}(z_i - z_k), r_{j,k}(z_j - z_k)] + [r_{i,j}(z_i - z_j), r_{i,k}(z_i - z_k)] = 0.$$

These are called the classical Yang-Baxter equations*.*

Remark. There are also *quantum Yang-Baxter equations*, which is of the form

$$R_{i,j}(z_i - z_j) = 1 + \hbar r(z_i - z_j) + O(\hbar^2),$$

and one can recover the classical equation from it.

Remark. We can naturally take $V_i \cong U(\mathfrak{g})$ to make a factorizable system, and in this case the condition in Proposition 19.2 is satisfied in $U(\mathfrak{g})^{\otimes 3}$.

Definition 19.2. *Classical r-matrices* are solutions of the classical Yang-Baxter equations which are meromorphic in a neighborhood of the origin with at least one point z where $r(z)$ is nondegenerate.

Definition 19.3. Two classical r-matrices $r(z)$, $\widetilde{r}(z)$ are *equivalent* if there is $\phi(x) \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\widehat{r}(z_1 - z_2) = (\phi(z_1) \otimes \phi(z_2))r(z_1 - z_2).$$

Theorem 19.3. *A classical r-matrix $r(z)$ extends to a meromorphic function on the complex plane satisfying $r_{1,2}(-z) = -r_{2,1}(z)$. Moreover, they fall into one of the following classes:*

1. Rational: $r(z) = c\Omega/z + P(z)$ *for some* $P \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathbb{C}[z]$ *and* $c \in \mathbb{C}^\times$*.*
2. Hyperbolic (or trigonometric): $r = f(e^{az})$*, where*

$$r(z) = \frac{P(e^{az})}{e^{a(n+1)z} - 1} + e^{-anz}Q(e^{az})$$

where P, Q *are polynomials and* $r(z + 2\pi i/a) = r(z)$*.*

3. Elliptic: $r(z + n\omega_1) = r(z + n\omega_2) = r(z)$*. This case exists only for Type A Lie algebras.*

Lecture 20

Mar. 30 — KZ Equations for $\mathfrak{sl}_2$

20.1 KZ Equations for $\mathfrak{sl}_2$

Remark. For this lecture, let $\mathfrak{g} = \mathfrak{sl}_2$. Then we have (letting $\kappa = k + h^\vee$)

$$\kappa \frac{\partial}{\partial z_i} \psi = \left(\sum_{j=1}^{N} \frac{\Omega_{i,j}}{z_i - z_j} \right) \psi, \quad i = 1, \dots, N.$$

Note that ψ takes values in $V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}$, where V_{μ_i} are the lowest weight modules (with height $-\mu_i$) Let $(V^{\mathfrak{n}_-})^{\underline{\lambda}}$ be the space of singular vectors, where $\underline{\lambda}$ is a weight. Write $\lambda = -\sum_{i=1}^N \mu_i + \mu$, where $\mu = \sum_{i=1}^r n_i \alpha_i$. Call $|\mu| = \sum_{i=1}^r n_i$ the *level* of the solution. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$. Let $(x, y) = \operatorname{tr}(xy)$, so that

$$(\beta, \gamma) = (\beta\omega, \gamma\omega) = \frac{\beta\gamma}{2}.$$

With this normalization, we have $h \to 1$, $\alpha \to 2$, $\rho = 1$, $k + h^\vee \to k + 2$, and $h = h^\vee = 2$.

Now we find the dimension of W. Let $p_N(m)$ be the number of ordered partitions of m into N parts, i.e.

$$p_N(m) = \#\left\{ \vec{m} = (m_1, \dots, m_N) \in \mathbb{Z}_+^N : \sum m_i = m \right\} = \binom{N+m-1}{m} = \frac{(N+m-1)!}{m!(N-1)!}.$$

Note that p_N has generating function $1/(1-q)^N$. Let v_i be the lowest weight vector in V_{μ_i}. Consider

$$e^{m_1} v_1 \otimes \cdots \otimes e^{m_N} v_N,$$

then we see that $\dim V^\lambda = p_N(m)$ where $\lambda = -\sum \mu_i + 2m$.

Proposition 20.1. *Let W be the space of singular vectors of weight $\lambda = -\sum \mu_i + 2m$ in V. Then for generic μ_i, we have*

$$\dim W = p_{N-1}(m) = \binom{N+m-2}{m}$$

Remark. In level 0, we have $W^0 = \mathbb{C}v$, where $v = v_1 \otimes \cdots \otimes v_N$. Then

$$\Omega_{i,j} v = \frac{1}{2}(v_1 \otimes \cdots \otimes hv_i \otimes \cdots \otimes hv_j \otimes \cdots \otimes v_N) = \frac{\mu_i \mu_j}{2} v.$$

Define the function

$$\psi_0(z_1, \dots, z_N) = \prod_{i<j} (z_i - z_j)^{\mu_i \mu_j / 2\kappa}.$$

Then $\psi_0 = \psi_0(z_1, \dots, z_n)v$ is a solution to the KZ equations.

We will now consider $N = 3$, which gives the simplest level 1 solution.

Proposition 20.2. *Any solution $\psi(z_1, z_2, z_3)$ of the KZ equations can be rewritten as*

$$\psi(z_1, z_2, z_3) = (z_1 - z_3)^{(\Omega_{1,2}+\Omega_{1,3}+\Omega_{2,3})/\kappa} f\left(\frac{z_1 - z_2}{z_1 - z_3}\right).$$

where f satisfies the equation

$$\kappa \frac{\partial}{\partial z} f(z) = \left(\frac{\Omega_{1,2}}{z} + \frac{\Omega_{2,3}}{z-1}\right) f(z). \tag{$*$}$$

Remark. Consider $L_0, L_{\pm 1}$ with $-d/dz$, $-zd/dz$, and $-z^2 d/dz$. Let

$$x = \frac{z_1 - z_2}{z_1 - z_3}, \quad y = z_1 - z_3, \quad t = z_1 + z_2 + z_3.$$

Then we have that

$$\kappa \frac{\partial \psi}{\partial x} = \left(\frac{\Omega_{1,2}}{x} + \frac{\Omega_{2,3}}{x-1}\right)\psi, \quad \kappa \frac{\partial \psi}{\partial y} = \frac{(\Omega_{1,2} + \Omega_{1,3} + \Omega_{2,3})}{y}\psi, \quad \kappa \frac{\partial \psi}{\partial t} = 0.$$

Then we see that $f = y^{-(\Omega_{1,2}+\Omega_{1,3}+\Omega_{2,3})/\kappa}\psi(x, y)$ does not depend on y.

Remark. Note that W is 2-dimensional on level 1: Assume $\mu_i \neq 0$, then

$$w_1 = \mu_1 e v_1 \otimes v_2 \otimes v_3 - \mu_2 v_1 \otimes e v_2 \otimes v_3$$
$$w_2 = \mu_3 v_1 \otimes e v_2 \otimes v_3 - \mu_2 v_1 \otimes v_2 \otimes e v_3$$

Theorem 20.1. *Any solution $f(z)$ of the equation $(*)$ can be written as follows:*

$$f(z) = z^{(\mu_1\mu_2 - 2\mu_1 - 2\mu_2)/2\kappa}(1-z)^{(\mu_2\mu_3)/2\kappa}\left(F(z)w_1 + z\frac{\kappa}{\mu_3}F'(z)w_2\right),$$

where $F(z)$ is a solution of the Gauss hypergeometric equation*:*

$$z(1-z)\frac{d^2}{dz^2}F = [c - (a+b+1)z]\frac{dF}{dt} - abF = 0,$$

where $a = \mu_3/\kappa$, $b = -\mu_1/\kappa$, and $c = 1 - (\mu_1 + \mu_2)/\kappa$.

Remark. The unique solution of the Gauss hypergeometric equation with $F(0) = 1$ is given by

$${}_2F_1(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n$$

in $|z| < 1$, where $(a)_n = a(a+1)\cdots(a+n-1)$.

Proposition 20.3. *For generic values of parameters,*

$$F^{(1)} = {}_2F_1(a, b, c, z) \quad \textit{and} \quad F^{(2)} = z^{1-c}{}_2F_1(a - c + 1, b - c + 1, 2 - c, z)$$

form a basis for the space of solutions.

20.2 Integral Formulas for Hypergeometric Equation

Remark. Let $\vec{z} = (z_1, \ldots, z_N)$, and define $\psi_1(\vec{z}, t) = \prod_{j=1}^N (t - z_j)^{-\mu_j/\kappa}$. Then we claim that

$$\psi_C(\vec{z}) = \psi_0(\vec{z}) \sum_{r=1}^N \int_C \psi_1(\vec{z}, t) \frac{dt}{t - z_r} e_r v$$

is a well-defined integral, where $v = v_1 \otimes \cdots \otimes v_N$ and $e_r = 1 \otimes \cdots \otimes e \otimes \cdots \otimes 1$ (r-th position).

Remark. Fix two distinct points $z_1, z_2 \in \mathbb{C}$. Consider a closed contour around z_1, z_2, which wraps around z_1 once in the negative direction, then z_1 once in the negative direction, then z_2 once in the positive direction, and finally z_1 once in the positive direction. This is called the *Pochhammer contour*. Using this, one can show that ψ_C is well-defined.

Proposition 20.4. *The function ψ_C as defined above is a W-valued solution of the KZ equations.*

Proof. This is a direct calculation. Define

$$T_i(\vec{z}, t) = \psi_1(\vec{z}, t) \frac{e_i}{t - z_i}, \quad Y(\vec{z}, t) = \sum_i T_i(\vec{z}, t), \quad H(\vec{z}, t) = \sum_i \frac{h_i + \mu_i}{t - z_i}.$$

We use the following lemma, which is also a direct calculation using commutators:

Lemma 20.1. *We have*

$$\kappa \frac{\partial}{\partial z_i} Y - \left[\sum_{j=1}^N \frac{\Omega_{i,j}}{z_i - z_j}, Y \right] = Y \frac{h_i + \mu_i}{t - z_i} - T_i H - \kappa \frac{\partial}{\partial t} T_i$$

Now we check that ψ_C is a solution. We can write

$$\psi_C(\vec{z}) = \psi_0(\vec{z}) \int_C Y(\vec{z}, t) v \, dt.$$

Since $\psi_0(\vec{z})v$ is a solution, it is enough to see that

$$\int_C \left[\kappa \frac{\partial}{\partial t_i} - \sum_{j=1}^N \frac{\Omega_{i,j}}{z_i - z_j}, Y \right] v \, dt = \int_C \kappa \frac{\partial}{\partial_t} T_i v \, dt = 0.$$

To check that ψ_C is W-valued, we want to show that $f \psi_C = 0$. Note that

$$[f, Y(\vec{z}, t)] = \psi_1(\vec{z}, t) \sum_i \frac{-h_i}{t - z_i},$$

so we get that

$$\begin{aligned} f \psi_C(\vec{z}, t) &= \psi_0(\vec{z}) \int_C \psi_1(\vec{z}, t) \sum_i \frac{-\mu_i}{t - z_i} v \, dt \\ &= \psi_0(\vec{z}) \int_C \psi_1(\vec{z}, t) \sum_i \frac{\mu_i}{t - z_i} v \, dt = -\psi_0(\vec{z}) v \int_C \kappa \frac{\partial}{\partial t} \psi_1(\vec{z}, t) \, dt = 0. \end{aligned}$$

This proves the desired result. $\square$

Example 20.0.1. Let $N = 3$. Then $z_1 = 0$, $z_2 = z$, and $z_3 = 1$. We have

$$\psi_C(0, z, 1) = z^{\mu_1\mu_2/2\kappa}(1-z)^{\mu_2\mu_3/2\kappa} \int_C t^{-\mu_1/\kappa}(t-z)^{-\mu_2/\kappa}(t-1)^{-\mu_3/\kappa}\left(\frac{ev_1 \otimes v_2 \otimes v_3}{t} + \frac{v_1 \otimes ev_2 \otimes v_3}{t-z} + \frac{v \otimes v_2 \otimes ev_3}{t-1}\right) dt.$$

Note that the function

$$F_C(z) = z^{(\mu_1+\mu_2)/\kappa} \int_C t^{-\mu_1/\kappa}(t-z)^{-\mu_2/\kappa}(1-t)^{-\mu_3/\kappa}\,\frac{dt}{t}$$

is a solution of the hypergeometric equation. Let $t = xz$, $z^{-1} \in [0, 1]$, and C be the Pochhammer contour around $0, 1$. Then we have that

$$\begin{aligned} F_C(z) &= \int_C x^{-1-\mu_1/\kappa}(1-x)^{-\mu_2/\kappa}(1-zx)^{-\mu_3/\kappa}\,dx \\ &= (1-e^{2\pi i\mu_1/\kappa})(1-e^{2\pi i\mu_2/\kappa}) \int_0^1 x^{-1-\mu_1/\kappa}(1-x)^{-\mu_2/\kappa}(1-zx)^{-\mu_3/\kappa}\,dx. \end{aligned}$$

For $\mathrm{Re}(\mu_i/\kappa) < 0$, $\mathrm{Re}(\mu_2/\kappa) < 1$, and $\mu_i/\kappa \notin \mathbb{Z}$, we have that

$$F(z) = \int_0^1 x^{-1-\mu_1/\kappa}(1-x)^{-\mu_2/\kappa}(1-zx)^{-\mu_3/\kappa}\,dx$$

is a solution of the hypergeometric equation.

Proposition 20.5. *We have*

$${}_2F_2(1)(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)} = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx.$$

20.3 Solutions in Higher Levels

Remark. Let $m = \mu/2 \in \mathbb{Z}_+$ be the level of the solution. Let

$$\vec{z} = (z_1, \dots, z_N), \quad \vec{t} = (t_1, \dots, t_m), \quad d\vec{t} = dt_1 \wedge \cdots \wedge dt_m.$$

Define the multi-valued differential form

$$\omega_{\vec{z},m} := \prod_{p<n}(t_p - t_n)^{2/\kappa} Y(\vec{z}, t_1)\cdots Y(\vec{z}, t_m)\,d\vec{t} \in Y_{\vec{z},m} := \{t \in \mathbb{C}^m : t_i \neq t_j, t_i \neq z_j\} \subseteq \mathbb{C}^m$$

Theorem 20.2. *The function*

$$\psi_C(\vec{z}) = \psi_0(\vec{z}) \int_C \omega_{\vec{z},m} v$$

is a solution of the KZ equations if C is a "legitimate" cycle.

Remark. Define the function

$$\psi_m(\vec{z}, t) = \prod_{i<j}(z_i - z_j)^{\mu_i\mu_j/2\kappa} \prod_{p,j}(t_p - z_j)^{-\mu_j/\kappa} \prod_{p<n}(t_p - t_n)^{2/\kappa},$$

and let (here Σ_m is the set of maps $s : \{1, \dots, m\} \to \{1, \dots, N\}$ such that $s^{-1}(j) = m_j$ for $1 \le j \le N$)

$$\rho_{\vec{m}}(\vec{z}, t) = \sum_{s \in \Sigma_m} \prod_{n=1}^{n} (t_n - z_{s(n)})^{-1},$$

where $\vec{m} = (m_1, \ldots, m_N)$ with $\sum m_i = m$. Then we have that

$$\psi_C(\vec{z}) = \left(\sum_{\vec{m}} \int_C \psi_m(\vec{z}, t) \rho_{\vec{m}}(\vec{z}, t)\, dt_1 \wedge \cdots \wedge dt_m \right) (e^{m_1} v_1 \otimes \cdots \otimes e^{m_N} v_N).$$

Lecture 21

Apr. 1 — Vertex Operators

21.1 KZ Equations in Higher Rank

Remark. Let $\mathfrak{g}$ be a simple Lie algebra. In the general setting, we are looking for $W = (V^{\mathfrak{n}_-})^\lambda$, where

$$V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_N},$$

$\lambda = -\sum_{i=1}^N \mu_i + \mu$ for $\mu \in \mathbb{Q}^+$. We call $\mu = \sum \ell_i \alpha_i$ the *multilevel* and $m = |\mu| = \sum \ell_i$ the *level* of the solution. Let $p_s = \sum_{j=1}^s \ell_j$ and $\nu : \{1, \dots, m\} \to \{1, \dots r\}$ (where r is the rank of $\mathfrak{g}$) such that $\nu(p) = s$ if $p_{s-1} < p \leq p_s$. Then we can define the function

$$\psi_\mu(\vec{z}, \vec{t}) = \sum_{1 \leq i < j \leq N} (z_i - z_j)^{\langle \mu_i, \mu_j \rangle / \kappa} \prod_{j=1}^N \prod_{p=1}^m (t_p - z_j)^{-\langle \alpha_{\nu(p)}, \mu_j \rangle / \kappa} \prod_{1 \leq p < n \leq m} (t_p - t_n)^{\langle \alpha_{\nu(p)}, \alpha_{\nu(n)} \rangle / \kappa}.$$

This is the multi-valued part, which we can combine with a rational part to give solutions to the KZ equations. The details can be found in the paper by V. Schechtman and A. Varchenko.

21.2 Vertex Operators

Remark. Let $\mathfrak{H} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \subseteq \widehat{\mathfrak{g}}$ be the extended Cartan subalgebra in $\widehat{g}$, which is called the *Heisenberg algebra*. For any $h, g \in \mathfrak{h}$, we have the commutation relation

$$[h[m], g[n]] = m \langle h, g \rangle \delta_{m,-n} c.$$

Let H_λ (for $\lambda \in \mathfrak{h}^*$) be the *Fock module* over $\mathfrak{H}$, which is generated by a highest weight vector v_λ with

$$h[n] v_\lambda = 0 \text{ for } n > 0, \quad h[0] v_\lambda = \lambda(h) v_\lambda, \quad c v_\lambda = v_\lambda.$$

We can decompose $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_0 \oplus \mathfrak{H}_-$, and $\mathfrak{H}_- = \mathfrak{h} \oplus t^{-1} \mathbb{C}[t^{-1}]$ acts freely on H_λ. Write

$$\mathfrak{H}_{\neq 0} = \mathfrak{H}_+ \oplus \mathfrak{H}_- \oplus \mathbb{C}c.$$

Note that H_λ does not notice λ, and we have

$$\begin{aligned} e^\mu : H_\lambda &\longrightarrow H_{\lambda+\mu} \\ v_\lambda &\longmapsto v_{\lambda+\mu}, \end{aligned}$$

which is an isomorphism of $\mathfrak{H}_{\neq 0}$-modules. Note that $e^\lambda e^\mu = e^{\lambda+\mu}$.

Define the *vertex operator* (for $z \in \mathbb{C}^\times, \mu \in \mathfrak{h}^*$)

$$X(\mu, z) = \exp\left(\sum_{n<0} \frac{h_\mu[n]}{-n} z^{-n}\right) \exp\left(\sum_{n>0} \frac{h_\mu[n]}{-n} z^{-n}\right) e^\mu z^{h_\mu[0]},$$

which acts via $H_\lambda \to \widehat{H}_{\lambda+\mu}$.

Definition 21.1. The *normal ordered product* $:X(\mu_1, z_1)X(\mu_2, z_2):$ is defined so that

$$:h_1[n]h_2[m]: = \begin{cases} h_1[n]h_2[m], & \text{if } m \geq 0, \\ h_2[m]h_1[n], & \text{if } m < 0, \end{cases} \quad \text{and} \quad :h[0]e^\mu: = :e^\mu h[0]: = e^\mu h[0].$$

Proposition 21.1. *If $|z_1| > |z_2|$, then $X(\mu_1, z_1)X(\mu_2, z_2)$ is a well-defined operator $H_\lambda \to \widehat{H}_{\lambda+\mu_1+\mu_2}$, and we have*

$$X(\mu_1, z_1)X(\mu_2, z_2) = (z_1 - z_2)^{(\mu_1,\mu_2)}:X(\mu_1, z_1)X(\mu_2, z_2):.$$

Proof. Note that by the Baker-Campbell-Hausdorff formula, we have $e^A e^B = e^B e^A e^{[A,B]}$ in this case. Let

$$M_2^- = \sum_{n<0} \frac{h_{\mu_2}[n]}{-n} z_2^{-n} \quad \text{and} \quad M_1^+ = -\sum_{n>0} \frac{h_{\mu_1}[n]}{-n} z_1^{-n}$$

Then we have

$$\begin{aligned}[M_1^+, M_2^-] &= -\sum_{m,n>0} \frac{[h_{\mu_1}[n], h_{\mu_2}[-m]]}{mn} z_1^{-n} z_2^m = -\sum_{n>0} \frac{n\langle h_{\mu_1}, h_{\mu_2}\rangle}{n^2} \left(\frac{z_2}{z_1}\right)^n \\ &= -\sum_{n>0} \frac{\langle \mu_1, \mu_2\rangle}{n} \left(\frac{z_2}{z_1}\right)^n = \langle \mu_1, \mu_2\rangle \log\left(1 - \frac{z_2}{z_1}\right),\end{aligned}$$

so we see that $e^{M_1^+} e^{M_2^-} = e^{M_2^-} e^{M_1^+} (1 - z_2/z_1)^{\langle\mu_1,\mu_2\rangle}$. $\square$

Corollary 21.0.1. *We have*

$$X(\mu_1, z_1) \cdots X(\mu_n, z_n) = \prod_{i<j} (z_i - z_j)^{(\mu_i,\mu_j)}:X(\mu_1, z_1) \cdots X(\mu_n, z_n):.$$

Proposition 21.2. *We have the following:*

1. $[h_\lambda[n], X(\mu, z)] = \langle \lambda, \mu\rangle z^n X(\mu, z)$;
2. *for $h(z) = \sum h[n] z^{-n-1}$ and $|z_1| > |z_2|$, we have*

$$h_\lambda(z_1)X(\mu, z_2) = :h_\lambda(z_1)X(\mu, z_2): + \frac{(\lambda, \mu)}{z_1 - z_2} X(\mu, z_2) \quad \textit{and} \quad \frac{d}{dz} X(\mu, z):h_\mu(z)X(\mu, z):.$$

Remark. Let g be a Lie algebra of type ADE. The *basic representation* of $\widehat{g}$ is $L_{0,1}$, and

$$L_{0,1} = \bigoplus_{\beta\in Q} H_\beta$$

by the Weyl-Kac character formula, where Q is the root lattice of g. Note that $\widehat{g}$ can be represented by $h[n]$ and e^μ (i.e. the action of $\widehat{g}$ is generated by $h[n]$ and e^μ).

Theorem 21.1. *Let $\mathfrak{g}$ be a Lie algebra of type ADE, so that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$. Let*

$$\varepsilon : Q \times Q \longrightarrow \{\pm 1\}$$

such that $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$, where Q is the root lattice of $\mathfrak{g}$. Then $L_{0,1} = \bigoplus_{\beta \in Q} H_\beta$ such that the action of $\widehat{g}$ on $L_{0,1}$ is given by

$$J_h(z) \sum_{n \in \mathbb{Z}} h[n] z^{-n-1}, \quad J_{e_\alpha}(z) = X(\alpha, z) C_\alpha, \quad J_{f_\alpha}(z) = X(-\alpha, z) C_{-\alpha},$$

where $J_x = \sum x[n] z^{-n-1}$ for $x \in \mathfrak{g}$ and $C_\alpha|_{H_\beta} = \varepsilon(\alpha, \beta)$.

Definition 21.2. Define $\mu' = \mu/\sqrt{\kappa}$ (where $\kappa = k + h^\vee$), and define a linear functional $v^*_{\mu'}$ on $H_{\mu'}$ by $(v^*_{\mu'}, v_{\mu'}) = 1$ and $(v^*_{\mu'}, v) = 0$ if v is a homogeneous vector of nonzero degree in $H_{\mu'}$.

Proposition 21.3. *We have*

$$\langle v^*_{\mu'}, X(\mu'_1, z_1) \cdots X(\mu'_N, z_N) v_{\mu'} \rangle = \prod_{i<j} (z_i - z_j)^{\langle \mu_i, \mu_j \rangle / \kappa} = \psi_0(z_1, \ldots, z_N),$$

where $\mu = \sum_{i=1}^n \mu_i$.

Corollary 21.1.1. *We have*

$$\psi_\mu(\vec{z}, \vec{t}) = \langle v^*_{-\lambda'}, X(-\alpha'_{\nu(1)}, t_1) \cdots X(-\alpha'_{\nu(m)}, t_m) X(\mu'_1, z_1) \cdots X(\mu'_N, z_N) v_0 \rangle,$$

where $\lambda = -\sum_{i=1}^N \mu_i + \mu$.

21.3 The β-γ System

Remark. Consider the *β-γ system B*, which is generated by β_n, γ_n, c for $n \in \mathbb{Z}$ with relations

$$[\beta_n, \gamma_m] = \delta_{n,-m} c, \quad [\beta_n, \beta_m] = [\gamma_n, \gamma_m] = 0, \quad [c, \beta_n] = [c, \gamma_n] = 0.$$

Define $\beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-n-1}$ and $\gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^{-n}$, and let

$$:\beta_n \gamma_m: = :\gamma_m \beta_n: = \begin{cases} \beta_n \gamma_m, & \text{if } n < 0, \\ \gamma_m \beta_n, & \text{if } n \geq 0. \end{cases}$$

The *Fock module* $H(B)$ over B is generated by v with $cv = v$, $\gamma_n v = 0$ for $n > 0$, and $\beta_n v = 0$ for $n \geq 0$.

Lecture 22

Apr. 6 — KZ Equations for $\mathfrak{sl}_2$, Part 2

22.1 The β-γ System, Continued

Remark. Let $z \in \mathbb{C}^\times$, then $\beta(z), \gamma(z)$ are well-defined operators $H(B) \to \widehat{H}(B)$. Moreover, $:\beta(z_1)\gamma(z_2):$ is also well-defined as an operator $H(B) \to \widehat{H}(B)$.

Proposition 22.1. *If $|z_1| > |z_2|$, then*

$$\beta(z_1)\gamma(z_2) = :\beta(z_1)\gamma(z_2): + \frac{c}{z_1 - z_2}.$$

Proof. We have

$$\beta(z_1)\gamma(z_2) - :\beta(z_1)\gamma(z_2): = \sum_{n \geq 0} [\beta_n, \gamma_{-n}] z_1^{-n-1} z_2^n = c \sum_{n \geq 0} z_1^{-n-1} z_2^n = \frac{c}{z_1 - z_2},$$

which is the desired result. □

Remark. Let H be a vector space and ω and antisymmetric bilinear form on H. Let $\widehat{H} = H \oplus \mathbb{C}c$ with

$$[h_1 + \lambda_1 c, h_2 + \lambda_2 c] = \omega(h_1, h_2)c.$$

Then we have $\widehat{H} = H^+ \oplus H^0 \oplus H^- \oplus \mathbb{C}c$, where H^+, H^- are isotropic subspaces with respect to ω and $H^0 = \ker \omega$. We can define relations

$$cu = u, \quad h^+ u = h^0 u = 0 \text{ for } h^+ \in H^+,\ h^0 \in H^0.$$

We call u the *vacuum vector.* Note that $\mathcal{F} = \mathbb{C}u \oplus H^- \mathcal{F}$, and if u^* is the linear functional with $\langle u^*, u \rangle = 1$, then u^* vanishes on $H^- \mathcal{F}$.

Theorem 22.1 (Wick). *Denote by $P(N)$ the set of partitions of $\{1, \dots, N\}$ into $N/2$ unordered pairs of numbers (with $P(N) = \varnothing$ for odd n). Let $h_1, \dots, h_N \in H$. Then*

$$\langle u^*, h_1 \cdots h_N u \rangle = \sum_{p \in P(N)} \prod_{n=1}^{N/2} \langle u^*, h_{p_1(n)} h_{p_2(n)} u \rangle = \sum_{p \in P(N)} \prod_{n=1}^{N/2} \omega(h_{p_1(n)}, h_{p_2(n)}),$$

where $p_1(n) < p_2(n)$ is the nth pair in p.

Remark. Consider a correlation function of the type

$$\rho_{\vec{m}}(\vec{z}, \vec{t}) = \frac{1}{m_1! \cdots m_N!} \langle u^*, \beta(t_1) \cdots \beta(t_m) \gamma(z_1)^{m_1} \cdots \gamma(z_N)^{m_N} u \rangle.$$

By Wick's theorem, this reduces to

$$\rho_{\vec{m}}(\vec{z},\vec{t}) = m_1! \cdots m_N! \sum_{s\in\Sigma_{\vec{m}}} \prod_{n=1}^{m} \langle u^*, \beta(t_n)\gamma(z_{s(n)})u\rangle,$$

where for $z \in \mathbb{C}^\times$ we have

$$\langle u^*, \beta(t)\gamma(z)u\rangle = \frac{1}{t-z} + \langle u^*, {:}\beta(t)\gamma(z){:}u\rangle = \frac{1}{t-z}.$$

22.2 Factorization of KZ Solutions for $\mathfrak{sl}_2$

Remark. Consider the case for $\mathfrak{g} = \mathfrak{sl}_2$. Let $U(t) = X(-\alpha', t)$, where α is the positive root of $\mathfrak{sl}_2$, and $Z_m(\lambda, z) = X(\lambda', z) \otimes \gamma^m(z)$. Then we have actions

$$\begin{tikzcd}[column sep=small] & H_\mu \otimes H(B) \arrow[dl, "U(t)"'] \arrow[dr, "Z_m(\lambda,z)"] & \\ \widehat{H}_{\mu-\alpha'} \otimes \widehat{H}(B) & & \widehat{H}_{\mu+\lambda'} \otimes \widehat{H}(B) \end{tikzcd}$$

Then we can write solutions to the KZ equations as

$$\psi_C(\vec{z}) = \sum_{\vec{m}} \int_C \langle v_{\lambda_0} \otimes u^*, U(t_1)\cdots U(t_m) Z_{m_1}(\lambda_1, z_1)\cdots Z_{m_N}(\lambda_N, z_N) v_0 \otimes u\rangle \, d\vec{t}$$

$$e^{m_1} m_1! v_1 \otimes \cdots \otimes \frac{e^{m_N}}{m_N!} v_N,$$

where $\lambda_0 = \sum \mu_i - 2m$.

22.3 Intertwining Operators via Fock Modules

Remark. Let α be the positive root for $\mathfrak{sl}_2$. Define an algebra A by

$$[\alpha_n, \alpha_m] = 2n\delta_{n,-m} c \quad \text{and} \quad [c, \alpha_n] = 0,$$

where $\alpha_n = h_\alpha[n]$, and define a module $H_\lambda(A)$ generated by v with

$$cv = v, \quad \alpha_n v = 0 \text{ for } n > 0, \quad \alpha_0 v = \lambda v \text{ for } \lambda \in \mathbb{C}.$$

Let $\mathfrak{h}^* \cong \mathbb{C}$ where $\alpha \mapsto 2$, so that

$$X(\mu, z) = X(\mu\alpha/2, z), \quad \mu \in \mathbb{C}.$$

Let $\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1}$ (compute $\alpha(z_1)\alpha(z_2)$ as an exercise for $|z_1| > |z_2|$). Then

$$J_e(z) = \beta(z), \quad J_h(z) = -2{:}\gamma(z)\beta(z){:} + \sqrt{\kappa}\alpha(z), \quad J_f(z) = -{:}\gamma(z)^2\beta(z){:} + \sqrt{\kappa}\alpha(z)\gamma(z) + k\frac{d\gamma(z)}{dz}$$

gives an action of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ on $H_\lambda(A) \otimes H(B)$.

Remark. Let $\beta = d/dx$, $\gamma = x$, then

$$e = \beta, \quad h = -2\gamma\beta + j, \quad f = -\gamma^2\beta + j\gamma$$

define a representation of the contragradient Verma module of highest weight j on $\mathbb{C}[z]$. There is then a notion of "affinization" which yields the above action for $\widehat{\mathfrak{sl}_2}$.

Theorem 22.2 (Wakimoto realization)**.** *Define an $\widehat{\mathfrak{sl}_2}$-action on $H_\lambda(A) \otimes H(B)$ as above. If λ, k are such that $M_{\sqrt{\kappa}\lambda,k}$ is irreducible, then this representation is isomorphic to $M_{\sqrt{\kappa}\lambda,k}$.*

Remark. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $\lambda, \lambda + \mu, k$ be generic. Then for all $m \geq 0$, there is a unique (up to scalar) intertwiner of level m

$$\Phi^m(z) : M_{\lambda,k} \longrightarrow \widehat{M}_{\lambda+\mu-2m,k} \otimes V_\mu(z),$$

where V_μ is the lowest weight Verma module. Writing $\mathcal{H}_{\lambda'} = H_\lambda(A) \otimes H(B)$, we have

$$\Phi^m(z) : \mathcal{H}_{\lambda'} \longrightarrow \mathcal{H}_{\lambda'+\mu'-2m'} \otimes V_\mu(z).$$

Lemma 22.1. *We have*

$$\Phi^0 w = X(\mu', z)e^{-\gamma(z)\otimes e}(w \otimes v_\mu),$$

where $w \in \mathcal{H}_{\lambda'}$ and v_μ is the lowest weight vector of $V_\mu(z)$.

Proof. Let $E(z) = \exp(-\gamma(z) \otimes e)$. Then one can compute that

$$[X(\mu', z)E(z), J_x^n \otimes 1]w \otimes v_\mu = z^n(1 \otimes x)X(\mu', z)E(z)w \otimes V_\mu$$

for all $x \in \mathfrak{sl}_2$ and $w \in \mathcal{H}_{\lambda'}$. $\square$

Lemma 22.2. *Let $U(t) = X(-\alpha', t)\beta(t)$, where $\alpha' = \alpha/\sqrt{\kappa}$. Then*

$$[J_e^n, U(t)] = 0 \quad \textit{and} \quad [J_f^n, U(t)] = \kappa\frac{d}{dt}(t^n X(-\alpha', t)).$$

Remark. Consider the expression

$$\int_C U(t)\, dt : \mathcal{H}_{\lambda'} \longrightarrow \widehat{\mathcal{H}}_{\lambda'-\alpha'}.$$

For generic λ, such an operator does not exist.

Remark. Define an operator

$$\begin{aligned} \Xi : \mathcal{H}_{\lambda'} &\longrightarrow \widehat{\mathcal{H}}_{\lambda'+\mu'-2m'} \otimes V_\mu \\ (z, t_1, \ldots, t_m) &\longmapsto X(\mu', z)e^{-\gamma(z)\otimes e}U(t_1)\cdots U(t_m)(w \otimes v_\mu), \end{aligned}$$

valid for $|z| > |t_1| > \cdots > |t_m| > 0$ (but can be extended to a larger region via analytic continuation).

Proposition 22.2. *We have*

$$\Xi(z, t_1, \ldots, t_m) = z^{\lambda\mu/\kappa}\prod_{j=1}^m (z - t_j)^{1-\mu/\kappa} t_j^{-\lambda/\kappa} \prod_{i<j}(t_i - t_j)^{2/\kappa}\Xi_0(z, t_1, \ldots, t_m),$$

where Ξ_0 is a meromorphic operator-valued function on $(\mathbb{C}^\times)^{m+1}$.

Proof. We have

$$e^{-\gamma(z)\otimes e}\beta(t) = -\frac{1\otimes e}{t-z}e^{-\gamma(z)\otimes e} + {:}e^{-\gamma(z)\otimes e}\beta(t){:} = {:}\exp(-\gamma(z)\otimes e)\left(\beta(t) - \frac{1\otimes e}{t-z}\right){:},$$

and thus we see that

$$e^{-\gamma(z)\otimes e}\beta(t_1)\cdots\beta(t_m) = {:}\exp(-\gamma(z)\otimes e)\left(\beta(t_1) - \frac{1\otimes e}{t_1-z}\right)\cdots\left(\beta(t_m) - \frac{1\otimes e}{t_m-z}\right){:},$$

which implies the result. $\square$

Lecture 23

Apr. 8 — Quantum Groups

23.1 Intertwining Operators via Fock Modules, Continued

Theorem 23.1. *We have (here $C = C_1 \times \cdots \times C_N$)*

$$\Phi^m(z) = \int_C \Xi(z, t_1, \ldots, t_m)\, dt_1 \wedge \cdots \wedge dt_m.$$

Remark. We also have

$$\int_C \frac{\partial}{\partial t_i} \Xi' \, d\vec{t} = 0.$$

Furethermore, $\langle \Phi(z_1) \cdots \Phi(z_N) \rangle$ reproduce the solutions of the KZ equations for $|z_1| > \cdots > |z_N|$.

Remark. Let $\Phi : V_{\lambda_1,k} \to \widehat{V}_{\lambda_2,k} \otimes V_\mu(z)$. A *Kazhdan-Lusztig (Moore-Seiberg) intertwiner* is

$$\widetilde{\Phi} : V_{\mu,k} \otimes V_{\lambda_1,k} \longrightarrow \widehat{V}_{\lambda_2,k}(z).$$

Here $V_{\mu,k}$ is a highest weight module for $\widehat{\mathfrak{g}}$ and $V_\mu(z)$ is a lowest weight evaluation module. Then we have $J^n \Phi(z) = \Phi(z) \Delta_{z,0}(J^n)$, where

$$\Delta_{z,0}(J^n) = \oint_z \frac{d\xi}{2\pi i} \xi^n \left(\sum_m (\xi - z)^{-m-1} y^m \right) \otimes 1 + 1 \otimes J^n.$$

Note that $\oint_0 J(z) = \sum_n J_n z^{-n-1}$. In vertex operator algebras, there is an expansion

$$A(\xi)B(z) = \sum \frac{A_n B(z)}{(\xi - z)^{n+1}},$$

and these correspond to shifting certain vector spaces "sitting at z" like above.

23.2 Hopf Algebras

Remark. A good reference for quantum groups is *A guide to quantum groups* by Chari and Pressley.

Definition 23.1. A *Hopf algebra* an algebra A with

- a product $m : A \otimes A \to A$,
- a coproduct $\Delta : A \to A \otimes A$,

- a unit $i : \mathbb{C} \to A$,
- a counit $\varepsilon : A \to \mathbb{C}$,
- an antipode $\gamma : A \to A$,

such that the following are satisfied:

1. properties of the product and coproduct:
   (a) $m(x \otimes m(y \otimes z)) = m(m(x \otimes y) \otimes z)$ (or more simply, $m(\mathrm{id} \otimes m) = m(m \otimes \mathrm{id})$);
   (b) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;
2. properties of the unit and counit:
   (a) $m(i(1) \otimes x) = m(x \otimes i(1)) = x$;
   (b) $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$;
3. properties of the antipode: $m((\mathrm{id} \otimes \gamma)(\Delta(x))) = m((\gamma \otimes \mathrm{id})(\Delta(x))) = i(\varepsilon(x))$;
4. consistency of product and coproduct: $\Delta(m(x \otimes y)) = m \otimes m(P_{2,3}(\Delta(x) \otimes \Delta(y)))$, where $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$ and
$$(xy)^{(1)} \otimes (xy)^{(2)} = m(x^{(1)} \otimes y^{(1)}) \otimes m(x^{(2)} \otimes y^{(2)});$$
5. consistency of unit and counit: $\varepsilon(i(1)) = 1$,
6. consistency of product and counit (and of coproduct and unit):
   (a) $\varepsilon(m(x \otimes y)) = \varepsilon(x)\varepsilon(y)$;
   (b) $\Delta(i(1)) = i(1) \otimes i(1))$;
7. consistency of product (and of coproduct) and antipode:
   (a) $\gamma(m(x \otimes y)) = m^{\mathrm{op}}(\gamma(x) \otimes \gamma(y))$, where $m^{\mathrm{op}}(x \otimes y) = m(y \otimes x)$;
   (b) $\Delta(\gamma(x)) = (\gamma \otimes \gamma)(\Delta^{\mathrm{op}}(x))$, where $\Delta^{\mathrm{op}}(x) = P_{1,2}(\Delta(x))$ and $P_{1,2} : x \otimes y \mapsto y \otimes x$;
8. consistency of unit (and of counit) and antipode:
   (a) $\gamma(i(1)) = i(1)$;
   (b) $\varepsilon(\gamma(x)) = \varepsilon(x)$.

Remark. We have the following:

1. Any Hopf algebra is an associative algebra with unit, and an associative coalgebra with counit.
2. To describe the Hopf structure for an associative algebra with unit, it is enough to describe the action and relations on generators. Moreover, The unit, counit, and antipode are uniquely determined by the multiplication and comultiplication.

Corollary 23.1.1. *We have the following:*

1. *If A is a finite dimensional Hopf algebra, then A^* is also a Hopf algebra with $m' = \Delta^*$, etc.*
2. *Let $\Delta \to \Delta^{\mathrm{op}}$ define another Hopf algebra. If A is commutative, then $A^{\mathrm{op}} \cong A$ with isomorphism γ.*
3. *If A, B are Hopf algebras, then $A \otimes B$ is also a Hopf algebra.*

Example 23.1.1. We have the following:

1. Let G be a group and $A = \mathbb{C}[G]$. Then we can define a Hopf algebra structure on A by
$$m(g \otimes h) = gh, \quad \Delta(g) = g \otimes g, \quad \gamma(g) = g^{-1}, \quad \varepsilon(g) = 1, \quad i(1) = e,$$
If G is finite, we can also define the dual Hopf algebra A^* to be $\mathcal{F}(G)$ (functions on G) with
$$m(f \otimes \phi)(g) = f(g)\phi(g), \quad \Delta(f)(g,h) = f(g,h), \quad i(1) = 1, \quad \varepsilon(f) = f(e), \quad \gamma(f)(g) = f(g^{-1}).$$

2. Let G be a linear algebraic group. Then $A = \mathcal{P}(G)$ (the space of polynomial functions on G) is a Hopf algebra with the same formulas as above for $\mathcal{F}(G)$ in the finite group case.

3. Let $\mathfrak{g}$ be a Lie algebra. Then $U(\mathfrak{g})$ is a Hopf algebra with
$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad \gamma(x) = -x.$$

Note that the examples (2) and (3) are dual to each other.

Definition 23.2. A *representation* of a Hopf algebra A is a left module over A as an associative algebra.

Example 23.2.1. We can take $x\lambda = \varepsilon(x)\lambda$, where λ is a $\mathbb{C}$-valued representation.

Remark. Let V be a finite-dimensional representation of A. Then there are two dual modules V^* and ${}^*V$ for A given by

- A acts on V^* by $af(v) = f(\gamma(a)v)$ for $v \in V$, $f \in V^*$;
- A acts on ${}^*V$ by $af(v) = f(\gamma^{-1}(a)v)$.

Exercise 23.1. Show that ${}^*V^*$ is canonically isomorphic to V, but ${}^{**}V$ and V^{**} are different in general.

Remark. Note that $V^* \otimes V \to \mathbb{C}$ and $\mathbb{C} \to V \otimes V^*$ are A-homomorphisms.

Remark. Let $\rho_i : A \to \mathrm{End}(V_i)$ for $i = 1, 2$. Define $\rho : A \to \mathrm{End}(V_1 \otimes V_2)$ by
$$\rho(x) = \rho_1 \otimes \rho_2(\Delta(x)).$$
Then $\Delta(\Delta \otimes I) = (I \otimes \Delta)\Delta$ implies associativity of the tensor product.

23.3 Quantum Groups

Remark. Let A be a generalized Cartan matrix with $d_i a_{i,j} = d_j a_{j,i}$. Fix $\{d_i\}$ to be positive integers with greatest common divisor 1. Then we get an extended Lie algebra $\mathfrak{g}_e$ with nondegenerate form $(\!|\cdot,\cdot|\!)$. Normalize the form $(\!|\cdot,\cdot|\!)$ so that
$$d_i = \frac{(\!|\alpha_i, \alpha_i|\!)}{2},$$
or equivalently, $(\!|\alpha, \alpha|\!) = 2$ for short roots α. Then $\mathfrak{h}_e \cong \mathfrak{h}_e^*$, and we can define
$$h_i = \frac{2\alpha_i}{(\!|\alpha_i, \alpha_i|\!)}.$$
Write $q = e^t$ for $t \in \mathbb{C}$ and $q^x = e^{tx}$ (assume that q is not a root of unity). Then $U_q(\mathfrak{g})$ is generated by e_i, f_i for $1 \le i \le r$ and q^h for $h \in \mathfrak{h}$, with relations

- $q^0 = 1$ and $q^{a+b} = q^a q^b$ for $a, b \in \mathfrak{h}$;
- $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ and $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$;
- $e_i f_j - f_j e_i = \delta_{i,j} \dfrac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}$;
- *q-Serre relations*: $\displaystyle\sum_{n=0}^{1-a_{i,j}} \frac{(-1)^n}{[n]_i![1-a_{i,j}-n]_i!} e_i^n e_j e_i^{1-a_{i,j}-n} = 0$, where

$$[n]_i = \frac{q^{nd_i} - q^{-nd_i}}{q^{d_i} - q^{-d_i}} \quad \text{and} \quad [n]_i! = \prod_{p=1}^{n} [p]_i,$$

  and identical relations hold for the f_i.

Note that we can formally write the q-Serre relations as $(\mathrm{ad}_{e_i}^{(q)})^{1-a_{i,j}}(e_j) = 0$, where

$$[x, y]_q = \mathrm{ad}_x^{(q)}(y) = xy - q^{(\!(\alpha,\beta)\!)} yx.$$

Lecture 24

Apr. 13 — Quantum Groups, Part 2

24.1 Quantum Groups, Continued

Remark. Recall the algebra $U_q(\mathfrak{g})$ defined last time. It is a Hopf algebra with coproduct

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes q^{d_i h_i} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + q^{-d_i h_i} \otimes f_i.$$

with counit $\varepsilon(e_i) = \varepsilon(f_i) = 0$ and $\varepsilon(q^h) = 1$. The antipode is given by

$$\gamma(e_i) = -e_i q^{-d_i h_i}, \quad \gamma(f_i) = -q^{d_i h_i} f_i, \quad \gamma(q^h) = q^{-h}.$$

There are two Hopf subalgebras $U_q(\mathfrak{g}^+)$ and $U_q(\mathfrak{g}^-)$ generated by $\{q^h, e_i\}$ and $\{q^h, f_i\}$, respectively.

24.2 Representations with Weight Decomposition

Remark. Suppose that we have a weight decomposition $V = \bigoplus_\mu V^\mu$, where $q^h v = q^{\mu(h)} v$ for $v \in V^\mu$ and $\mu \in \mathfrak{h}^*$. Similarly as before, we have the Verma module

$$M_\lambda^{(q)} = \operatorname{Ind}_{U_q(\mathfrak{g}^+)}^{U_q(\mathfrak{g})} X_\lambda,$$

where X_λ is the 1-dimensional module where all e_i act as 0. We can also define $L_\lambda^{(q)} = M_\lambda^{(q)}/I_\lambda$, where I_λ is the maximal proper submodule of $M_\lambda^{(q)}$.

Theorem 24.1. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and $q \in \mathbb{C}$ (not a root of unity). Then*

1. *$L_\lambda^{(q)}$ is finite-dimensional if and only if $\lambda \in P^+$, and in this case, $\dim L_\lambda^{(q)} = \dim L_\lambda$ (the same is true for the dimensions of the weight subspaces);*
2. *the category $C(\mathfrak{g}, q)$ of finite-dimensional $U_q(\mathfrak{g})$-modules is semisimple.*

Remark. The contragradient module for $M = \bigoplus_\lambda M^\lambda$ is given by the restricted dual $M^c = \bigoplus_\lambda (M^\lambda)^*$, where we have

$$\langle g v^*, v \rangle = \langle v^*, \tau(g) v \rangle, \quad v^* \in M^c, v \in M, g \in U_q(\mathfrak{g}).$$

Here $\tau(e_i) = q^{d_i h_i} f_i$, $\tau(f_i) = e_i q^{-d_i h_i}$, and $\tau(q^h) = q^h$. Note that $\tau(ab) = \tau(b)\tau(a)$, this is done so that $(M_1 \otimes M_2)^c \cong M_1^c \otimes M_2^c$ as $U_q(\mathfrak{g})$-modules. Note that the $U_q(\mathfrak{g})$-morphism

$$M_\lambda^{(q)} \longrightarrow (M_\lambda^{(q)})^c$$

is generally not an isomorphism, its image is $L_\lambda^{(q)}$.

Remark. Let $\rho \in \mathfrak{h}^*$ such that $(\!|\rho, h_i|\!) = 1$, i.e. $\alpha_i(\rho) = d_i$. Then

$$\gamma^2(x) = q^{2\rho} x q^{-2\rho}.$$

Let V^* be the dual module of V. Then a priori, V^{**} has a different $U_q(\mathfrak{g}_e)$-module structure from V.[1]

Proposition 24.1. *If V is a finite-dimensional module over $U_q(\mathfrak{g}_e)$ with weight decomposition, then*

$$\begin{aligned} \delta_V : V &\longrightarrow V^{**} \\ v &\longmapsto q^{2\rho} v \end{aligned}$$

is an isomorphism of $U_q(\mathfrak{g}_e)$-modules.

24.3 Quasitriangular Structure

Definition 24.1. Let A be a Hopf algebra and $R = \sum_i a_i \otimes b_i \in A \otimes A$ invertible such that[2]

$$R\Delta(x) = \Delta^{\mathrm{op}}(x) R, \quad (\Delta \otimes I)R = R_{1,3}R_{2,3}, \quad (I \otimes \Delta)R = R_{1,3}R_{1,2}.$$

Such an R is called a *universal R-matrix* for A, and A with a choice of R is called *quasitriangular*.

Remark. Note that if V_1, V_2 are two finite-dimensional representations of A, then $V_1 \otimes V_2 \cong V_2 \otimes V_1$. Thus we can define $\check{R}_{V_1,V_2} = P_{1,2} R|_{V_1 \otimes V_2}$, where $P_{1,2}$ is the isomorphism $V_1 \otimes V_2 \to V_2 \otimes V_1$ given by permuting the factors.

Proposition 24.2. *We have the following:*

$$(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1_A, \quad (\gamma \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \gamma^{-1})(R) = R^{-1}, \quad (\gamma \otimes \gamma)(R) = R.$$

Moreover, $R^{\mathrm{op}} = P(R)$ (where $P(a \otimes b) = b \otimes a$) is a universal R-matrix for A^{op}.

Definition 24.2. A *braided tensor category* is an additive category $\mathcal{C}$ with a distinguished object 1, a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and isomorphisms

$$\begin{gathered} \alpha_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3), \\ \lambda_V : 1 \otimes V \cong V, \quad \rho_V : V \otimes 1 \cong V, \\ \sigma_{V_1,V_2} : V_1 \otimes V_2 \cong V_2 \otimes V_1 \end{gathered}$$

satisfying the following properties:

1. *triangle axiom:* the following diagram commutes:

$$\begin{tikzcd} (X \otimes 1) \otimes Y \arrow[rr, "\rho_X \otimes \mathrm{id}"] \arrow[dr, "\alpha_{X,1,Y}"'] & & X \otimes Y \\ & X \otimes (1 \otimes Y) \arrow[ur, "\mathrm{id} \otimes \lambda_Y"'] & \end{tikzcd}$$

2. *pentagon axiom:* all maps $((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \to V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))$ obtained by composing α and α^{-1} are equal;

[1] Recall that $\mathfrak{g}_e$ is the extended Lie algebra with an added derivation d, which gives a non-degenerate bilinear form $(\!|\cdot,\cdot|\!)$.
[2] Here $R_{i,j}$ denotes R acting in the i, j factors of the tensor product $A \otimes A \otimes A$.

3. *hexagon axioms:* the isomorphisms $V_1 \otimes (V_2 \otimes V_3) \to (V_2 \otimes V_3) \otimes V_1$ given by $\sigma_{V_1, V_2 \otimes V_3}$ and
$$\alpha^{-1}(\mathrm{id}_{V_2} \otimes \delta_{V_1,V_3})\alpha(\delta_{V_1,V_2} \otimes \mathrm{id}_{V_3})\alpha^{-1}$$
are equal, and similarly for σ^{-1}.

Theorem 24.2. *Let A be a quasitriangular Hopf algebra. The category of representations of A is a braided tensor category with $1 = \mathbb{C}$, trivial unit morphisms and associative morphisms, and $\delta_{V,W} = \check{R}_{V,W}$.*

Remark. In this category, every object V has a dual V^*, and we have morphisms $1 \to V \otimes V^*$ and $V^* \otimes V \to 1$. This is known as *rigidity.*

24.4 Quantum Yang-Baxter Equation

Proposition 24.3 (Quantum Yang-Baxter equation)**.** *We have $R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$.*

Proof. We can write
$$R_{1,2}R_{1,3}R_{2,3} = R_{1,2}(\Delta \otimes \mathrm{id})R = (\Delta^{\mathrm{op}} \otimes \mathrm{id})(R)R_{1,2} = P_{1,2}(R_{1,3}R_{2,3})R_{1,2} = R_{2,3}R_{1,3}R_{1,2},$$
which is the desired formula. □

Definition 24.3. Define the *braid group on N strands* to be the group B_N generated by $b_1, \ldots, b_{N-1}$ with relations $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ and $b_i b_j = b_j b_i$ for $|i - j| > 1$.

Theorem 24.3. *Let A be a quasitriangular Hopf algebra and $V_1, \ldots, V_N$ representations of A. Define*
$$\check{R}_i = V_1 \otimes \cdots \otimes V_N \longrightarrow V_1 \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_N$$
by $\check{R}_i = P_{i,i+1} R_{V_i, V_{i+1}}$. Then $\check{R}_i \check{R}_{i+1} \check{R}_i = \check{R}_{i+1} \check{R}_i \check{R}_{i+1}$.

Corollary 24.3.1. *In a braided tensor category, we can define isomorphisms*
$$\beta^{\pm}_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow Y \otimes (X \otimes Z)$$
by $\beta^+_{X,Y,Z} = \alpha(\sigma_{X,Y} \otimes \mathrm{id}_Z)\alpha^{-1}$ and $\beta^-_{X,Y,Z} = \alpha(\sigma^{-1}_{X,Y} \otimes \mathrm{id}_Z)\alpha^{-1}$. Let
$$\beta^{\pm}_{1,2} = \beta^{\pm}_{X,Y,Z} : X \otimes (Y \otimes (Z \otimes U)) \longrightarrow Y \otimes (X \otimes (Z \otimes U)),$$
$$\beta^{\pm}_{2,3} = \mathrm{id} \otimes \beta^{\pm}_{Y,Z,U} : X \otimes (Y \otimes (Z \otimes U)) \longrightarrow X \otimes (Z \otimes (Y \otimes U)).$$
Then we have $\beta^{\pm}_{1,2}\beta^{\pm}_{2,3}\beta^{\pm}_{1,2} = \beta^{\pm}_{2,3}\beta^{\pm}_{1,2}\beta^{\pm}_{2,3}$ and $\beta^{\pm}_{X,Y,Z}\beta^{\mp}_{Y,X,Z} = \mathrm{id}$.

24.5 Quantum Double

Remark. Let A be a finite-dimensional Hopf algebra. Let $\{a_i\}$ a basis for A, and $\{a^i\}$ the dual basis of A^*. Define $D(A) = A \otimes (A^*)^{\mathrm{op}}$. We want a Hopf algebra structure on $D(A)$ so that the natural embeddings $A \hookrightarrow D(A)$ and $(A^*)^{\mathrm{op}} \hookrightarrow D(A)$ are Hopf algebra morphisms.

One constructs the *quantum double* (or *Drinfeld double*) as follows. Let the structure constants of
$$A \otimes A \otimes A \longrightarrow A$$

(note that $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$) be $m^i_{j,k,\ell}$, i.e. $a_j a_k a_\ell = \sum_i m^i_{j,k,\ell} a_i$, and the structure constants of

$$A \longrightarrow A \otimes A \otimes A$$

(note that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$) be $\mu_i^{j,k,\ell}$. Then we can define multiplication on $D(A)$ by relations

$$a_s a^t = \mu_s^{k,j,n} m^t_{p,\ell,k} \sigma^p_n a^\ell a_j$$

(with implied summations), which uniquely defines a Hopf algebra structure on $D(A)$.

Theorem 24.4. *The Hopf algebra $D(A)$ is quasitriangular, with*

$$R = \sum_i (a_i \otimes 1_{A^*}) \otimes (1_A \otimes a^i).$$

Example 24.3.1. Let G be a finite group and $A = \mathbb{C}[G]$. Then $A^* = \mathcal{F}(G)$, and we have

$$D(A) = \mathbb{C}[G] \ltimes \mathcal{F}(G).$$

Lecture 25

Apr. 15 — Quantum Groups, Part 3

25.1 Quantum Double for $U_q(\mathfrak{g}_e)$

Remark. We construct the quantum double for $U_q(\mathfrak{g}_e)$. First consider $U_q(\mathfrak{g}_e^{\pm})$, generated by $\{q^h, e_i\}$ and $\{q^h, f_i\}$, respectively:

$$U_q(\mathfrak{g}_e^{\pm}) = \mathbb{C}[\mathfrak{h}_e] \otimes U_q(\mathfrak{n}^{\pm}),$$

where $\mathbb{C}[\mathfrak{h}_e]$ is the group algebra associated with q^h. Then we can define the restricted dual (by degree)

$$U_q(\mathfrak{g}^{\pm})^* = \mathbb{C}[\mathfrak{h}_e^*] \otimes U_q(\mathfrak{n}^{\pm})^*,$$

where the pairing is $\langle q^h, q^\lambda \rangle = q^{\lambda(h)}$ for $\lambda \in \mathfrak{h}_e^*$.

Theorem 25.1. *There is an isomorphism of Hopf algebras*

$$\theta : U_q(\mathfrak{g}_e^-) \longrightarrow (U_q(\mathfrak{g}_e^+)^*)^{\mathrm{op}}$$

such that $\theta(q^h) = q^h$ for all $h \in \mathfrak{h}_e$, where we identify $\mathfrak{h}_e$ with $\mathfrak{h}_e^*$ using $(\!|\cdot,\cdot|\!)$.

Example 25.0.1. Check that Theorem 25.1 does not hold for $q = 1$ (one algebra is commutative whereas the other is not).

Theorem 25.2. *Theorem 25.1 implies that we can identify*

$$D(U_q(\mathfrak{g}_e^+)) = U_q(\mathfrak{g}_e^+) \otimes U_q(\mathfrak{g}_e^-),$$

which has a two-sided ideal H generated by $q^h \otimes 1 - 1 \otimes q^h$.

Theorem 25.3. *The map given by*

$$e_i \otimes 1 \mapsto e_i, \quad 1 \otimes f_i \mapsto f_i, \quad q^h \otimes 1 \mapsto q^h, \quad 1 \otimes q^h \mapsto q^h$$

defines an isomorphism $\phi : (U_q(\mathfrak{g}_e^+) \otimes U_q(\mathfrak{g}_e^-))/H \to U_q(\mathfrak{g}_e)$.

Remark. The algebra $D(U_q(\mathfrak{g}_e^+))$ has R-matrix given by

$$R = q^{\sum_p x_p \otimes x_p} \sum_i a_i \otimes a^i = \sum_{n \geq 0} \frac{t^n (\sum_p x_p \otimes x_p)}{n!} \sum_i a_i \otimes a^i,$$

where $\{a_i\}$ is a basis in $U_q(\mathfrak{n}^+)$ and $\{a^i\}$ is a basis in $U_q(\mathfrak{n}^-)$.

Example 25.0.2. For $U_q(\mathfrak{sl}_2)$, we have

$$R = q^{\frac{1}{2} h \otimes h} \sum_{n \geq 0} q^{n(n-1)/2} \frac{(q - q^{-1}) e^n \otimes f^n}{[n]!},$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$. For general $\mathfrak{g}_e$, the formula is given by Khoroshkin-Tolstoy.

Theorem 25.4. *Let $C(\mathfrak{g}_e, q)$ be the category of finite-dimensional representations with weight decomposition of $U_q(\mathfrak{g}_e)$ (with q not a root of unity). Then $C(\mathfrak{g}_e, q)$ has the structure of a braided tensor category with $\sigma_{V,W} = \check{R}_{V,W} = P_{1,2}R_{V,W}$ and unity associativity morphism.*

25.2 Quantum Casimir Element

Remark. Let A be a quasitriangular Hopf algebra, and define

$$u = m((\gamma \otimes \mathrm{id}_A)(R^{\mathrm{op}})) = \sum_i \gamma(b_i)a_i,$$

where $R = \sum_i a_i \otimes b_i$. Then $u \in A$ is well-defined (i.e. u actually lies in A) if A is finite-dimensional. In the case where $A = U_q(\mathfrak{g}_e)$, the element is still defined as an operator in any finite-dimensional representation.

Proposition 25.1. *We have the following:*

1. $\gamma^2(x) = uxu^{-1}$ *for all* $x \in A$;
2. $u^{-1} = m((\gamma^{-1} \otimes \mathrm{id}_A)(R^{\mathrm{op}}))$;
3. $\Delta(u) = (u \otimes u)(R^{\mathrm{op}}R)^{-1} = (R^{\mathrm{op}}R)^{-1}(u \otimes u)$.

Remark. Recall that we have defined $q^{2\rho} \in U_q(\mathfrak{g}_e)$ satisfying $\gamma^2(x) = q^{2\rho}xq^{-2\rho}$.

Theorem 25.5. *Let $A = U_q(\mathfrak{g}_e)$ and $C = q^{2\rho}u^{-1}$. Then C is a central element and is well-defined for any finite-dimensional representation. Moreover, C satisfies*

$$\Delta(C) = (R^{\mathrm{op}}R)(C \otimes C) = (C \otimes C)(R^{\mathrm{op}}R)$$

and $C|_V = q^{(\!|\lambda, \lambda+2\rho|\!)}\,\mathrm{id}_V$, where V is the highest weight module with highest weight λ.

Corollary 25.5.1. *Let V_λ, V_μ, V_ν be highest weight modules over $U_q(\mathfrak{g}_e)$ with corresponding highest weights λ, μ, ν, respectively. Then for any embedding of $U_q(\mathfrak{g}_e)$-modules $V_\lambda \subseteq V_\mu \otimes V_\nu$, we have*

$$\check{R}^2|_{V_\lambda \subseteq V_\mu \otimes V_\nu} = q^{c(\lambda)-c(\mu)-c(\nu)}\,\mathrm{id}_{V_\lambda},$$

where $c(\lambda) = (\!|\lambda, \lambda + 2\rho|\!)$.

Proof. Apply $\check{R}^2 = \Delta(C)(C^{-1} \otimes C^{-1})$ (which follows from Theorem 25.5) to $C|_V = q^{(\!|\lambda,\lambda+2\rho|\!)}\,\mathrm{id}_V$. $\square$

25.3 Intertwining Operators

Remark. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $C(\mathfrak{g}, q)$ the category of finite-dimensional representations of $U_q(\mathfrak{g})$ with weight decomposition. Then all morphisms in this category can be described using $\mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda^{(q)}, V)$. Let

$$H_{V,\mu}^\lambda = \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda^{(q)}, V \otimes L_\mu^{(q)}).$$

Let V, W be finite-dimensional representations. Then

$$\bigoplus_\mu H_{V,\mu}^\lambda \otimes H_{W,\nu}^\mu \cong \mathrm{Hom}_{U_q(\mathfrak{g})}(L_\lambda^{(q)}, V \otimes W \otimes L_\nu^{(q)})$$

via the composition $\Phi_1 \otimes \Phi_2 \mapsto (1 \otimes \Phi_2) \circ \Phi_1$.

Remark. Let $\check{R}^+ = PR$ and $\check{R}^- = P(R^{\mathrm{op}})^{-1} = (\widehat{R}^+)^{-1}$. We can act on an intertwiner

$$\Phi : L_\lambda^{(q)} \longrightarrow V \otimes W \otimes L_\nu^{(q)}$$

by multiplying by $\check{R}^\pm_{V,W} \otimes \mathrm{id}_{L_\nu^{(q)}}$, which gives another intertwiner $L_\lambda^{(q)} \to W \otimes V \otimes L_\nu^{(q)}$. This gives

$$\check{B}^\pm_{W,V}(\lambda,\nu) : \bigoplus_\mu H^\lambda_{V,\mu} \otimes H^\mu_{W,\nu} \xrightarrow{\cong} \bigoplus_{\mu'} H^\lambda_{W,\mu'} \otimes H^{\mu'}_{V,\nu}.$$

We call such $\check{B}^\pm$ *exchange matrices.*

Proposition 25.2. *The exchange matrices satisfy*

1. $\check{B}^\pm_{1,2}\check{B}^\pm_{2,3}\check{B}^\pm_{1,2} = \check{B}^\pm_{2,3}\check{B}^\pm_{1,2}\check{B}^\pm_{2,3}$, *where both sides are viewed as isomorphisms*

$$\mathrm{Hom}(L_\lambda^{(q)}, V \otimes W \otimes U \otimes L_\nu^{(q)}) \to \mathrm{Hom}(L_\lambda^{(q)}, U \otimes W \otimes V \otimes L_\nu^{(q)});$$

2. $\check{B}^\pm\check{B}^\mp = \mathrm{id}$.

Remark. Write the associativity morphism as $\alpha : (V \otimes L_\nu^{(q)}) \otimes W \cong V \otimes (L_\nu^{(q)} \otimes W)$. Then we have

$$\alpha_{V,W}(\lambda,\mu) : \bigoplus_\mu H^\mu_{V,\nu} \otimes H^\lambda_{W,\mu} \xrightarrow{\cong} \bigoplus_{\mu'} H^\lambda_{V,\mu'} \otimes H^{\mu'}_{W,\nu}.$$

For $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$, let V, W be highest weight modules with highest weights i, j. Then $\alpha_{V,W}$ depends on $i, j, \lambda, \nu, \mu, \mu'$, and the elements are called the *quantum* $6j$*-symbols* (or *Racah coefficients*).

Remark. The hexagon axiom gives the diagram

$$\begin{tikzcd}
V \otimes (L_\nu^{(q)} \otimes W) \arrow[r, "\alpha^{-1}"] \arrow[d, "\mathrm{id}\otimes\check{R}_{L_\nu^{(q)},W}"'] & (V \otimes L_\nu^{(q)}) \otimes W \arrow[d, "\check{R}_{V\otimes L_\nu^{(q)},W}"] \\
V \otimes (W \otimes L_\nu^{(q)}) \arrow[r, "\beta^+_{L_\nu^{(q)},V,W}"] & W \otimes (V \otimes L_\nu^{(q)})
\end{tikzcd}$$

Thus we can write $\check{B}_{V,W}(\lambda,\nu) = \check{R}_2\alpha^{-1}_{V,W}(\lambda,\nu)\check{R}_1^{-1}$, where

$$\check{R}_1 : \bigoplus_{\mu'} H^{\mu'}_{W,\nu} \longrightarrow \bigoplus_\mu H^\mu_{V,\nu} \quad \text{and} \quad \check{R}_2 : \bigoplus_\mu H^\lambda_{W,\mu} \longrightarrow \bigoplus_{\mu'} H^\lambda_{W,\mu'}.$$

For $U_q(\mathfrak{sl}_2)$, we have $\check{B} = D_2\check{\alpha}D_1$, where D_1, D_2 are diagonal.

Remark. Let λ, μ, ν be generic, so that $L_\lambda^{(q)} \cong M_\lambda^{(q)}$, and V, W finite-dimensional. Then we can describe $H^\lambda_{V,\mu}$ explicitly using the map

$$\begin{aligned} \langle\cdot\rangle : \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda^{(q)}, M_\mu^{(q)} \otimes V) &\longrightarrow V^{\lambda-\mu} \\ \Phi &\longmapsto \langle\Phi\rangle = (v_\mu^* \otimes \mathrm{id}_V)(\Phi v_\lambda), \end{aligned}$$

where v_λ is a highest weight vector in $M_\lambda^{(q)}$ and v_μ^* is a lowest weight vector in $(M_\mu^{(q)})^*$. This gives

$$H^\lambda_{V,\mu} = \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda^{(q)}, V \otimes M_\mu^{(q)}) \cong V^{\lambda-\mu}.$$

Thus for any generic λ and $v \in V^{\lambda-\mu}$, there is a unique operator

$$\Phi_\lambda^v : M_\lambda^{(q)} \longrightarrow M_\mu^{(q)} \otimes V$$

such that $\Phi_\lambda^v(v_\lambda) = v \otimes v_\mu +$ lower order terms. Similarly, if $v \in V$ and $w \in W$ are homogeneous vectors, we can define

$$\Phi_\lambda^{v\otimes w} = (\mathrm{id} \otimes \Phi^w) \circ \Phi_\lambda^v : L_\lambda^{(q)} \longrightarrow V \otimes W \otimes L_\nu^{(q)},$$

where $\nu = \lambda + \mathrm{weight}(v) + \mathrm{weight}(w)$. Then we have

$$(\breve{R}_{V,W}^{\pm} \otimes \mathrm{id})\Phi_\lambda^{v_i \otimes w_j} = \sum_{i,j} \Phi_\lambda^{w_{j'} \otimes v_{i'}} B_{i',j'}^{i,j}(\lambda)^{\pm}.$$

Using this, we can define $\breve{B}_{V,W}^{\pm}(\lambda, \nu) : (V \otimes W)^{\lambda-\nu} \cong (W \otimes V)^{\lambda-\nu}$ between weight subspaces. Summing over ν, we get an operator

$$B_{V,W}^{\pm}(\lambda) : V \otimes W \longrightarrow W \otimes V$$

which preserves weights. The $B_{V,W}^{\pm}(\lambda)$ satisfy *dynamical Yang-Baxter equations.*

Lecture 26

Apr. 20 — Geometric Aspects

26.1 Local Systems

Remark. Let X be a real manifold, and let $\Omega^i(X)$ denote the space of differential i-forms on X. Let

$$\Omega^i(X, E) = E \otimes \Omega^i(X)$$

denote the space of differential forms with values in a vector bundle E.

Definition 26.1. A *connection* on E is a map $\nabla : \Gamma(U, E) \to \Omega^1(U, E)$ satisfying

$$\nabla(fs) = f\nabla s + s \otimes df$$

for a function f and a section s.

Remark. We can define $\nabla_\xi : E \to E$ by $s \mapsto \langle \nabla s, \xi \rangle$ with

$$\nabla_\xi(fs) = f\nabla_\xi s + (\partial_\xi f)s.$$

In local coordinates, we can write

$$\nabla_i = \nabla_{\partial/\partial x_i} = \partial_{x_i} + A_i(x).$$

If $U_{\alpha,\beta}$ are the transition functions for E, then

$$U_{\alpha,\beta}(d + A_\alpha)U_{\alpha,\beta}^{-1} = \nabla_\beta = d + A_\beta.$$

This implies $A_\beta = U_{\alpha,\beta} A_\alpha U_{\alpha,\beta}^{-1} + U_{\alpha,\beta} dU_{\alpha,\beta}^{-1}$, and such a map is called a *gauge transformation.* Note that we get a *curvature 2-form* by defining $F = dA + A \wedge A$, and we have

$$F(\xi, \eta) = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}.$$

The components of the 2-form F are given by

$$[\partial_i + A_i, \partial_j + A_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j] = F_{i,j}.$$

Definition 26.2. A *flat* connection is a connection with 0 curvature, i.e.

$$[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi,\eta]},$$

or equivalently, $F_{i,j} = 0$. A *local system* is a vector bundle which admits a flat connection.

Remark. A connection ∇ gives rise to *parallel transport*. Given a curve $\gamma : [0,1] \to X$ and $v \in E_{\gamma(0)}$ (the fiber of E at $\gamma(0)$), there is a unique section s of E along γ such that $\nabla_{\dot\gamma} s = 0$ and $s(\gamma(0)) = v$. There is a *holonomy operator* $M_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ given by $s(\gamma(0)) \mapsto s(\gamma(1))$.

Theorem 26.1. *If ∇ is a flat connection, then M_γ depends only on the homotopy class of γ.*

Corollary 26.1.1. *The map $\gamma \mapsto M_\gamma$ (with fixed basepoint) gives a representation of the fundamental group in E_{x_0}.*

Remark. Building on Corollary 26.1.1, there is in fact a bijection between local systems and finite-dimensional representations of $\pi_1(X, x_0)$.

Definition 26.3. Let $U \subseteq X$ and s a section of E over U. We say that s is *flat* if $\nabla_\xi s = 0$ for all $\xi \in U$. We denote the space of flat sections over U by $\Gamma_f(U, E)$.

Lemma 26.1. *Let $U \subseteq X$ be open and simply connected. Then for any $p \in U$, there is an isomorphism*

$$\begin{aligned} \Gamma_f(U, E) &\longrightarrow E_p \\ s &\longmapsto s(p). \end{aligned}$$

In other words, the sheaf of flat sections is locally constant.

Remark. Let E, E' be two vector bundles over X with connections ∇, ∇'. Then

$$\Delta(s \otimes s') = (\nabla s \otimes s') + (s \otimes \nabla' s')$$

defines a connection on $E \otimes E'$. Similarly, if E^* is dual to E with pairing $\langle s, s^* \rangle$, then

$$\langle \nabla_\xi s, s^* \rangle + \langle s, \nabla_\xi s^* \rangle = \partial_\xi \langle s, s^* \rangle$$

defines a connection on E^*. One can also define a pullback bundle f^*E and corresponding connection, given a map $f : Y \to X$ and vector bundle E on X.

26.2 Cohomology of Local Systems

Remark. We have a sequence

$$0 \longrightarrow E \xrightarrow{\nabla} \Omega^1(X, E) \xrightarrow{\nabla} \Omega^2(X, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n(X, E) \longrightarrow 0$$

where $\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes d\omega$. However, this is not a complex in general.

Proposition 26.1. *If ∇ is a flat connection, then $\nabla^2 = 0$ and $\Omega^\bullet(X, E)$ is a complex.*

Remark. If ∇ is flat, then we can consider the de Rham cohomology $H^\bullet_{\mathrm{dR}}(X, E)$ with coefficients in a local system (the cohomology of the complex $\Omega^\bullet(X, E)$).

Remark. Let $\sigma : \Sigma_k \to X$ be a singular k-simplex. Consider $\sum_j \varepsilon_j \sigma_j$, where ε_j is a flat section of $\sigma^* E$ over Σ_k. Then we can define a boundary map ∂ by

$$\partial(\varepsilon_j \sigma_j) = \sum_j \sum_\ell \varepsilon_j^\ell \sigma_j^\ell (-1)^\ell,$$

where ε_j^ℓ is the restriction of the section to the ℓth face and σ_j^k is the ℓth face of σ_j. This gives rise to the *singular homology* $H_\bullet(X, E)$ and *singular cohomology* $H^\bullet(X, E)$.

Remark. Let E be a local system with fiber V. Let $\omega \in \Omega^k(X, E)$ and $\sigma : \Sigma_k \to X$, so $\sigma^*\omega$ is a differential form on Σ_k with coefficients in the local system σ^*E. If

$$\omega \in \Omega^k(\Sigma_k, \sigma^*E) = \Omega^k(\Sigma_k) \otimes V$$

and $C = \varepsilon\sigma$ (where ε is the coefficient section of the dual bundle), then we can define

$$\int_C \omega = \langle \varepsilon, \int_{\Sigma_k} \sigma^*\omega \rangle.$$

Theorem 26.2 (Stokes theorem)**.** *We have*

$$\int_C d\omega = \int_{\partial C} \omega,$$

and integration defines a pairing $B : H^\bullet(X, E^*) \otimes H^\bullet_{\mathrm{dR}}(X, E) \to \mathbb{C}$.

Theorem 26.3 (De Rham theorem)**.** *If X is a smooth real manifold and B is nondegenerate, then* $H^\bullet(X, E^*) \cong H^\bullet_{\mathrm{dR}}(X, E)$.

Remark. Consider an *affine complex manifold*, i.e. a complex manifold X which is isomorphic to a closed subvariety in $\mathbb{C}^N$. We can write $X = \mathbb{C}^N \setminus D$ for some subvariety $D \subseteq \mathbb{C}^N$ of codimension 1.

Theorem 26.4 (Complex de Rham theorem)**.** *Let X be an affine complex manifold and $H^\bullet_{\mathrm{dR}}(X, E)$ be the cohomology of the global holomorphic forms. Then B is nondegenerate and $H^\bullet(X, E^*) \cong H^\bullet_{\mathrm{dR}}(X, E)$.*

Corollary 26.4.1. *For a complex affine manifold of complex dimension n, we have that $H_i(X, E)$ and $H^i(X, E)$ are nonzero only for $i \leq n$.*

26.3 Hyperplane Arrangements

Remark. Consider a finite set S of codimension 1 affine hyperplanes in $\mathbb{C}^n$ (a *hyperplane arrangement*), and let X be the complement of its union. Let $H \in S$ and ℓ_H the corresponding linear functional (i.e. H is the set of solutions to $\ell_H(\vec{z}) = 0$). Let $\vec{\lambda} = \{\lambda_H\}_{H \in S}$ for some constants $\lambda_H \in \mathbb{C}$ and define

$$\psi(\vec{z}) = \prod_{H \in S} \ell_H(\vec{z})^{\lambda_H}.$$

Definition 26.4. For any S and $\vec{\lambda}$, denote by $\mathcal{L}_{\vec{\lambda}}$ the 1-dimensional local system over X with local sections $\psi(\vec{z})f(\vec{z})$ (where f is a holomorphic function on X) and connection $\nabla_i = \partial_i$. Equivalently, we can define $\mathcal{L}_{\vec{\lambda}}$ to be the trivial line bundle with connection

$$\nabla_i f = \partial_i f + f\partial_i(\log \psi) = \partial_i f + f\psi^{-1}\partial_i\psi.$$

Definition 26.5. Define the *Orlik-Solomon algebra* $\mathrm{OS}^\bullet(X)$ to be the graded exterior algebra generated by $d(\log \ell_H(\vec{z}))$, which can be described by generators and relations.

Remark. For $f \in \mathrm{OS}^\bullet(X)$, we call ψf a *hypergeometric form*, and denote the space of hypergeometric forms by $\Omega^\bullet_{\mathrm{hg}}(X, \mathcal{L}_{\vec{\lambda}}) \subseteq \Omega^\bullet(X, \mathcal{L}_{\vec{\lambda}})$.

Theorem 26.5. *For almost all $\vec{\lambda}$ (including $\vec{\lambda} = 0$), the map*

$$\Omega^\bullet_{\mathrm{hg}}(X, \mathcal{L}_{\vec{\lambda}}) \longrightarrow \Omega^\bullet(X, \mathcal{L}_{\vec{\lambda}})$$

is a quasi-isomorphism.

Example 26.5.1. Let $X = \mathbb{C} \setminus \{0\}$ and $\psi(z) = z^\lambda$. Then

$$z^\lambda d(\log z) = z^\lambda \frac{dz}{z}$$

is a hypergeometric 1-form, and z^λ is a 0-form. Note that

$$d(z^\lambda) = \lambda z^\lambda \frac{dz}{z}.$$

We have $H^1 = \mathbb{C}$ for $\lambda = 0$ and $H^1 = 0$ for almost all $\lambda \neq 0$.

Exercise 26.1. Show that in Example 26.5.1, we have $H^1 = \mathbb{C}$ if $\lambda \in \mathbb{Z}$ and $H^1 = 0$ if $\lambda \notin \mathbb{Z}$. So we see that Theorem 26.5 holds for $\lambda \notin \mathbb{Z} \setminus \{0\}$.

Remark. One can consider discriminantal arrangements on configuration spaces. Let $\vec{z} = (z_1, \ldots, z_N) \in \mathbb{C}^N$ such that $z_i \neq z_j$. For $m \geq 0$, we have a map $X_{m+N} \to X_N$ given by forgetting all but the first N components. This map is a fiber bundle. Denote the fiber over $\vec{z} \in X_N$ by $Y_{\vec{z},m}$, i.e.

$$Y_{\vec{z},m} = \{\vec{t} = (t_1, \ldots, t_m) \in \mathbb{C}^m : t_i \neq t_j,\, t_i \neq z_p\}.$$

Moreover, there is a natural action of S_N on X_N and of S_m on $Y_{\vec{z},m}$ by permuting coordinates. Let

$$Q = \{q_{p,n}, r_{p,j} : 1 \leq p < n \leq m,\, 1 \leq j \leq N\}.$$

Then we can define

$$\psi_{\vec{z}}(\vec{t}) = \prod_{p<n} (t_p - t_n)^{q_{p,n}} \prod_{p,j} (t_p - z_j)^{r_{p,j}}.$$

If $q_{p,n}, r_{p,j}$ are symmetric with respect to S_m, then this defines a local system $\overline{\mathcal{L}}_Q$ on $\overline{Y}_{\vec{z},m} = Y_{\vec{z},m}/S_m$. Moreover, we have $H^\bullet(\overline{Y}_{\vec{z},m}, \overline{\mathcal{L}}_Q) = (H^\bullet(Y_{\vec{z},m}, \mathcal{L}_Q))^{S_m}$.

Remark. Consider the KZ equations for $\mathfrak{sl}_2$. Let $\vec{z} = (z_1, \ldots, z_N)$, $\vec{t} = (t_1, \ldots, t_m)$, and $\kappa \in \mathbb{C}^\times$. Let $\mathcal{L} = \mathcal{L}(\vec{\mu}, \kappa)$. Then recall that we have

$$\psi_{\vec{m}}(\vec{z}, \vec{t}) = \prod_{i<j} (z_i - z_j)^{\mu_i \mu_j / 2\kappa} \prod_{p,j} (t_p - z_j)^{-\mu_j/\kappa} \prod_{p<n} (t_p - t_n)^{2/\kappa},$$

where $\prod_{i<j} (z_i - z_j)^{\mu_i \mu_j / 2\kappa}$ is constant on every $Y_{\vec{z},m}$.

Lecture 27

Apr. 22 — Geometric Aspects, Part 2

27.1 Relation to KZ Equations

Remark. Recall that for $V = M_{\mu_1} \otimes \cdots \otimes M_{\mu_N}$ and $z_1, \ldots, z_N$, we have

$$\psi_C = \sum_{\vec{m}} \left(\int_C \psi_m(\vec{z}, \vec{t}) \rho_m(\vec{z}, \vec{t})\, d\vec{t} \right) v_{\vec{m}},$$

where $\vec{m} = (m_1, \ldots, m_N)$ with $\sum m_i = m$ and $v_{\vec{m}} = e^{m_1} v_1 \otimes \cdots \otimes e^{m_N} v_N$. Note that $\dim Y_{\vec{z},m} = m$. Also, the above integral depends only on the class of $C \in C_m(Y_{\vec{z},m}, \mathcal{L}^*)$.

Theorem 27.1 (Aomoto)**.** *Let $m = 1$. Then for almost all μ, κ,*

1. $H_0(Y_{\vec{z},1}, \mathcal{L}^*) = 0$*;*
2. $\dim H_1(Y_{\vec{z},1}, \mathcal{L}^*) = \dim H^1(Y_{\vec{z},1}, \mathcal{L}) = N - 1$*;*
3. *the forms $\psi(\vec{z}, t) dt/(t - z_j)$ for $j = 1, \ldots, N$ for a basis in $H^1(Y_{\vec{z},1}, \mathcal{L})$;*
4. *the Pochhammer loops C_i around z_i, z_N form a basis in $H_1(Y_{\vec{z},1}, \mathcal{L}^*)$.*

Corollary 27.1.1. *For every $\vec{z}$ and almost all μ_i, κ, the map $C \mapsto \psi_C$ is an isomorphism between $H_1(Y_{\vec{z},1}, \mathcal{L}^*)$ and the space of W-valued solutions of the KZ equations, where W is the space of singular vectors of weight $-\sum \mu_i + 2$.*

Theorem 27.2 (Arbitrary level)**.** *For all $\vec{z}$ and almost all $\vec{\mu}, \kappa$,*

1. $H_i(Y_{\vec{z},m}, \mathcal{L}^*) = 0$ *for $i < m$;*
2. $\dim H_m(Y_{\vec{z},m}, \mathcal{L}^*) = \dim H^m(Y_{\vec{z},m}, \mathcal{L}) = m! p_{N-1}(m)$*, where $p_k(m)$ is the number of partitions of m into ordered sums of k nonnegative integers.*

Remark. Taking the smaller space $H_m(Y_{\vec{z},m}, \mathcal{L}^*)^{S_m}$, gets rid of the $m!$ in Theorem 27.2(2). This gives a natural generalization of Corollary 27.1.1 for arbitrary m: We have

$$H_m(Y_{\vec{z},m}, \mathcal{L}^*)^{S_m} \cong W,$$

where W is the space of singular vectors of weight $-\sum m_i + 2m$.

27.2 Gauss-Manin Connection

Remark. Consider the fiber bundle $\pi : X_{m+N} \to X_N$ with fiber $Y_{\vec{z},m}$. Recall that there is an action of S_N on X_N. We have can choose a cycle in every fiber, but we want to match them in a compatible way.

Note that $\psi_m(\vec{z}, \vec{t})$ defines a local system on X_{m+N}, which gives local systems on $Y_{\vec{z},m}$ by restriction. Thus we get a vector bundle $H_m(X_{m+N}/X_N, \mathcal{L})$ with a fiber at $\vec{z}$ given by $H_m(Y_{\vec{z},m}, \mathcal{L})$. Note that by analytic continuation we may move $\vec{z} \to \vec{z}'$, so for any $\vec{z} \in X_N$, a choice of $C \in H_i(Y_{\vec{z},m}, \mathcal{L})$ defines a cycle in each $Y_{\vec{z}',m}$ defines a cycle in each $Y_{\vec{z}',m}$ for z' close enough to z. This defines a connection in

$$C(X_{m+N}/X_N, \mathcal{L})$$

via the condition that all local sections produced in this way are flat (the *Gauss-Manin connection*).

Theorem 27.3. *For all μ_i and almost all κ, the map $C \mapsto \psi_C$ gives an isomorphism of local systems*

$$H_m(X_{m+N}/X_N, \mathcal{L}(\mu, \kappa))^{S_m} \cong W_{\mathrm{KZ}},$$

where the left-hand side is endowed by the Gauss-Manin connection and the right-hand side is the trivial vector bundle over X_N with fiber $W = (V^{\mathfrak{n}^-})^\lambda$ ($\lambda = -\sum \mu_i + 2m$), endowed with the KZ connection.

Remark. Theorem 27.3 fails for $k \in \mathbb{Z}_+$ (recall that $\kappa = k + h^\vee$).

27.3 Relative Homology

Example 27.0.1. Let $X = \mathbb{C} \setminus \{0, 1\}$ and E the 1-dimensional local system on X defined by

$$\psi(z) = z^\alpha (1-z)^\beta$$

for $\alpha, \beta \in \mathbb{C}$. Then for generic α, β, we have $H_1(X, E^*) \cong \mathbb{C}$, generated by the Pochhammer loop. For $\operatorname{Re}\alpha \gg 0$ and $\operatorname{Re}\beta \gg 0$, we can write

$$\int_C f(z)\psi(z)\,dz = (1 - e^{2\pi i\alpha})(1 - e^{2\pi i\beta}) \int_0^1 f(z)\,\psi(z)\,dz.$$

We call this integral on the right-hand side the *regularization* of $[0, 1]$, and denote it by $[0, 1]_{\mathrm{reg}}$.

Definition 27.1. Let $X = \overline{X} \setminus D$, where $\overline{X}$ is a real manifold and $D \subseteq \overline{X}$ is a closed subset. The space $C_k(\overline{X}, D, E)$ of *relative chains* is the space of finite linear combinations

$$\sum \varepsilon_i \sigma_i,$$

where ε_i is a flat section of E over $\sigma_i(\Sigma^k) \cap X$ and σ_i is a singular k-simplex in $\overline{X}$.

Remark. Note that $C_k(X, E) \subseteq C_k(\overline{X}, D, E)$, which gives a map

$$H_k(X, E) \longrightarrow H_k(\overline{X}, D, E), \tag{$*$}$$

though this map is generally not injective or surjective.

Example 27.1.1. Note that $[C] = (1 - e^{2\pi i\alpha})(1 - e^{2\pi i\beta})[0, 1]$ in Example 27.0.1, and in fact the map on homology is surjective for all $\alpha, \beta \in \mathbb{Z}$. This is not true for standard homology, e.g. consider $\alpha = \beta = 0$.

Theorem 27.4. *For a general hyperplane arrangement and general λ_H, the map $(*)$ is an isomorphism.*

Definition 27.2. Call any subset L of the set of all hyperplanes S such that $\bigcap_{H \in L} H \neq \varnothing$ an *edge*, and denote $\lambda_L = \sum_{H \in L} \lambda_H$.

Theorem 27.5. *If for every edge L, we have $\lambda_L \notin \mathbb{Z}$, then the map $(*)$ is an isomorphism.*

Theorem 27.6. *Let $D_{\vec{z},m}$ be the union of the hyperplanes $\{t_i = t_j, t_i = z_k\}$, so that $Y_{\vec{z},m} = \mathbb{C}^m \setminus D_{\vec{z},m}$. Then for all μ_i, κ generic, we have:*

1. *$H_i(\mathbb{C}^m, D_{\vec{z},m}, \mathcal{L}(\vec{\mu}, \kappa)) = 0$ for $i < m$.*
2. *For $\vec{z} \in \mathbb{R}^N$, define*
$$Y^{\mathbb{R}}_{\vec{z},m} = \{\vec{t} \in \mathbb{R}^m : t_i \neq t_j, t_i \neq z_k\}$$
Let $\{D_i\}$ be the set of bounded connected components of $Y^{\mathbb{R}}_{\vec{z},m}$ (so each D_i is the interior of some convex polytope), and fix an arbitrary section of $\mathcal{L}(\vec{\mu}, \kappa)$ over each D_i. Then
$$H_m(\mathbb{C}^m, D_{\vec{z},m}, \mathcal{L}(\vec{\mu}, \kappa)) = \bigoplus \mathbb{C}[D_i],$$
and thus $H_i(Y_{\vec{z},m}, \mathcal{L}(\vec{\mu}, \kappa)) = 0$ for $i < m$.

Remark. We can pick another basis for the homology from $\vec{m} = (m_1, \ldots, m_{N-1})$ (with $m_i \geq 0$) and a bijection ϕ which assigns to a pair $1 \leq a, b \leq N-1$ an index $i = \phi(a,b) \in \{1, \ldots, m\}$ ($m = \sum m_i$).

Theorem 27.7. *Let $z_1, \ldots, z_N$ such that $[z_i, z_N]$ do not intersect except for at z_N, and define*
$$D_{\vec{m},\phi} = \left\{ t_1, \ldots, t_m : \begin{array}{c} t_i = z_N + u^a_b(z_a - z_N) \text{ if } i = \phi(a,b), \\ 0 < u^1_1 < u^1_2 < \cdots < u^1_{m_1} < 1, \\ \vdots \\ 0 < u^{N-1}_1 < \cdots < u^{N-1}_{m_{N-1}} < 1 \end{array} \right\} \subseteq Y_{\vec{z},m}.$$
Then for almost all μ_i, κ, the $D_{\vec{m},\phi}$ form a basis for the relative homology.

27.4 Monodromy of KZ Equations

Remark. Recall the KZ equations:
$$\kappa \frac{\partial}{\partial z_i} \psi = \left(\sum_{j=1, j \neq i}^{N} \frac{\Omega_{i,j}}{z_i - z_j} \right) \psi, \quad \kappa \notin \mathbb{Q}.$$
We can interpret this system as a flat connection in a trivial bundle with fiber $V_1 \otimes \cdots \otimes V_N$ over
$$X_N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N : z_i \neq z_j\}.$$
The solutions are then given by the flat sections, which we denote by $\Gamma_f(U, V_{\mathrm{KZ}})$.

Proposition 27.1. *For every γ, the map $M_\gamma : V \to V$ is a $\mathfrak{g}$-homomorphism and M_γ preserves singular vectors. If $\vec{z}_0 = (z_1, \ldots, z_N)$ and $\gamma(0) = \gamma(1) = \vec{z}_0$, then $\gamma \to M_\gamma$ is a representation of $\pi_1(X_N, \vec{z}_0)$.*

Theorem 27.8 (Artin)**.** *We have the following:*

1. *$\pi_1(X_N) = PB_N$, the* pure braid group *defined by $PB_N = \ker \sigma$ where $\sigma : B_N \to S_N$;*
2. *S_N acts on X_N by permuting coordinates, and $\pi_1(X_N / S_N) = B_N$.*

Remark. The map $B_N \to \pi_1(X_N/S_N)$ in Theorem 27.8(2) is defined as follows: Fix $\vec{z}_0 = (z_1, \ldots, z_N) \in \mathbb{R}^N$ such that $z_N < z_{N-1} < \cdots < z_2 < z_1$, and for each $\sigma \in S_N$ let
$$V^\sigma = V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(N)}.$$
Let $\sigma : V \to V^\sigma$ be defined by $v_1 \otimes \cdots \otimes v_N \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(N)}$. Fix a basepoint and lift a loop γ

in X_N/S_N to a path in X_N. We have an operator

$$\widetilde{M}_\gamma = \sigma M_\gamma : V \longrightarrow V^\sigma,$$

and we can define $\breve{M}_i^{\pm}(z) = (\widetilde{M}_{\gamma_i})^{\pm 1}$ for some path γ_i.

Theorem 27.9. *We have the following:*

1. $\breve{M}_i^{\pm}\breve{M}_i^{\mp} = \mathrm{id}$;
2. $\breve{M}_i^{\pm}\breve{M}_{i+1}^{\pm}\breve{M}_i^{\pm} = \breve{M}_{i+1}^{\pm}\breve{M}_i^{\pm}\breve{M}_{i+1}^{\pm}$.

Corollary 27.9.1. *If all representations $V_i \cong W$ are isomorphic and $z_1 > \cdots > z_N$, then $b_i \mapsto \breve{M}_i^{\pm}(\vec{z})$ is a representation of the braid group B_N on $V = W^{\otimes N}$.*

Lecture 28

Apr. 27 — Geometric Aspects, Part 3

28.1 Asymptotics of Solutions to KZ Equations

Remark. Let $z_1 \gg z_2 \gg \cdots \gg z_N$, and let

$$D = \{(z_1, \ldots, z_N) \in \mathbb{R}^N : z_1 > \cdots > z_N\} \subseteq X_N$$

Make a change of variables (note that $u_i > 0$)

$$u_1 = z_1 - z_2, \quad u_2 = \frac{z_2 - z_3}{z_1 - z_2}, \quad \ldots, \quad u_{N-1} = \frac{z_{N-1} - z_N}{z_{N-2} - z_{N-1}}, \quad u_N = z_1 + \cdots + z_N.$$

Let $\vec{z}(t) \to 0$ as $t \to 0$ be a path, then $z_i(t)/z_{i+1}(t) \to \infty$ for $i = 1, \ldots, N-1$, and $u_i(t) \to 0$.

Definition 28.1. Let f be a smooth vector-valued function on D. We say that

$$\lim_{z_1 \gg \cdots \gg z_N} f(z) = v$$

if $\lim_{u_i \to 0} f(u) = v$. We write $f \sim \phi(z)v$ at $z_1 \gg \cdots \gg z_N$ if

$$f(z) = \phi(z)(v + o(z)),$$

where $o(z)$ vanishes in the limit $u_i \to 0$.

Lemma 28.1. *We have the following*

$$\kappa \frac{\partial}{\partial u_i} = \frac{\Omega_i + \cdots + \Omega_{N-1}}{u_i} + \mathrm{Reg} \quad \textit{and} \quad \frac{\partial}{\partial u_N}\psi = 0,$$

where Reg *is regular in u_i around 0. Here $\Omega_i = \sum_{j>i} \Omega_{i,j}$, and the Ω_i commute with each other.*

Proposition 28.1. *Let $\kappa \notin \mathbb{Q}$.*

1. *For every vector $v \in V$ (the space where solutions take values) which is a common eigenvector for Ω_i with eigenvalues ω_i, there exists a unique solution ψ_v such that*

$$\psi_v \sim \prod_i u_i^{(\omega_i + \cdots + \omega_{N-1})/\kappa} v.$$

2. *If $\{v_a\}$ forms a basis in V consisting of eigenvectors of Ω_i, then ψ_a forms a basis in the space of solutions. In particular, we have an isomorphism*

$$\begin{aligned} \phi_{z_1 \gg \cdots \gg z_N} : \Gamma_f(D, V_{\mathrm{KZ}}) &\xrightarrow{\cong} V \\ \psi_a &\longmapsto v_a. \end{aligned}$$

We call $(D, \phi : \Gamma_f(D, V_{\mathrm{KZ}}) \to V)$ an asymptotic zone.

Remark. Let $f_i : L_{i-1} \to V_i \otimes L_i$. Then,

$$F = f_{N-1} \cdots f_1 : L_0 \longrightarrow V_1 \otimes \cdots \otimes V_N,$$

and the $F(v)$ (ranging over all choices of $v \in L_0$ and f_i) form a basis in $V_1 \otimes \cdots \otimes V_N$.

Theorem 28.1. *Let $v \in L_0$ and f_i as above. Let $f_i^* : L_i^* \to L_{i-1}^* \otimes V_i$ be the adjoint operator, and define*

$$\Phi_i = \widehat{\Phi}^{f_i^*} : V_{\lambda_i^*,k} \longrightarrow V_{\lambda_{i-1}^*,k} \otimes V_i(z_i).$$

Then $\psi(z_1, \ldots, z_{N-1}) = \langle v, \Phi_1 \cdots \Phi_{N-1} \cdot \rangle \in V_1 \otimes \cdots \otimes V_N$ is an asymptotic solution with vector $F(v)$.

28.2 Monodromy of KZ Equations, Continued

Remark. We have an isomorphism $\phi_\sigma : \Gamma(\sigma(D), V_{\mathrm{KZ}}) \to V$ for any $\sigma \in S_N$. Define the *monodromy operator* as the following composition:

$$M_\sigma^\pm(\infty) : V \xrightarrow{\cong} \Gamma_f(D, V_{\mathrm{KZ}}) \xrightarrow{\cong} \Gamma(\sigma(D), V_{\mathrm{KZ}}) \xrightarrow{\cong} V.$$

Let $\breve{M}_i^\pm(\infty) = \sigma_i M_i^\pm : V \to V^{\sigma_i}$.

Lemma 28.2. *Let $A(\vec{z}) = \prod_{i=1}^{N-1} (z_i - z_N)^{-\Omega_i/\kappa}$. Then*

$$\breve{M}_i(\infty) = \lim_{z_1 \gg \cdots \gg z_N} A(\sigma_i(\vec{z})) \breve{M}_i(\vec{z}) A^{-1}(\vec{z}).$$

Remark. Let V be a finite-dimensional $\mathfrak{g}$-module and $\lambda, \mu \in P^+$. Let

$$\widetilde{H}_{\mu,V}^\lambda = \mathrm{Hom}_{\mathfrak{g}}(L_\lambda, L_\mu \otimes V).$$

For every $g \in \widetilde{H}_{\mu,V}^\lambda$, we get an intertwiner $\widehat{\Phi}^g : V_{\lambda,k} \to V_{\mu,k} \otimes V(z)$. Define

$$\widehat{\Phi}^{g_1 \otimes g_2} = (\widehat{\Phi}^{g_1} \otimes \mathrm{id}) \circ \Phi^{g_2} : V_{\lambda,k} \longrightarrow V_{\nu,k} \otimes V(z_1) \otimes W(z_2)$$

for $g_1 \in \widetilde{H}_{\nu,V}^\mu$ and $g_2 \in \widetilde{H}_{\mu,W}^\lambda$. Let $g = g_1 \otimes g_2$, then we can define $A^\pm \widehat{\Phi}^g(z_1, z_2)$ by analytic continuation.

Theorem 28.2. *We have the following:*

1. *The operators $A^\pm \widehat{\Phi}^g(z_1, z_2)$ are well-defined for $z_1 > z_2 > 0$ and are $\widehat{g}$-intertwiners.*
2. *For $z_2 > z_1 > 0$, we have $A^\pm \widehat{\Phi}^g(z_1, z_2) = P_{V,W} \widehat{\Phi}^{\breve{B}^\pm g}(z_2, z_1)$, where*

$$\breve{B}_{V,W}^\pm(\nu^*, \lambda^*) : \mathrm{Hom}(L_\nu^*, V \otimes W \otimes L_\lambda^*) \longrightarrow \mathrm{Hom}(L_\nu^*, W \otimes V \otimes L_\lambda^*).$$

28.3 Equivalence of Categories

Remark. In 3 variables z_1, z_2, z_3, we have the equation

$$\kappa \frac{\partial \psi}{\partial x} = \left(\frac{\Omega_{1,2}}{x} + \frac{\Omega_{2,3}}{x-1} \right) \psi,$$

where $x = (z_1 - z_2)/(z_1 - z_3)$. We have the following asymptotic zones:

- $z_1 \gg z_2 \gg z_3$, corresponding to $x \to 1$;
- $z_2 \gg z_1 \gg z_3$, corresponding to $x \to -\infty$;
- $z_1 - z_3 \gg z_1 - z_2 \gg 0$, corresponding to $x = 0$.

We can define the *Drinfeld associator* $\Phi_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$ by the composition

$$V \xrightarrow{\phi^{-1}_{z_1-z_3 \gg z_1-z_2 \gg 0}} \Gamma_f(D, V_{\mathrm{KZ}}) \xrightarrow{\phi_{z_1 \gg z_2 \gg z_3}} V$$

This is a nontrivial isomorphism.

Theorem 28.3. *Let $\kappa \notin \mathbb{Q}$. Then the category $C(\mathfrak{g})$ of finite-dimensional representations of $\mathfrak{g}$ is a braided tensor category with $\alpha_{V_1,V_2,V_3} = \Phi_{V_1,V_2,V_3}$ and $\sigma_{V_1,V_2} = Pe^{\pi i \Omega/\kappa}$.*

28.4 Quantum Affine Algebras

Remark. Recalled that in the classical case, the KZ equations had solutions taking values in $V_1(z_1) \otimes \cdots \otimes V_N(z_N)$, which are representations of $\mathfrak{g}[z, z^{-1}]$. One can similarly construct

$$V_1^q(z_1) \otimes \cdots \otimes V_N^q(z_N),$$

which are representations of the *quantum affine algebra* $U_q(\widehat{\mathfrak{g}})$, defined similarly as before. We then have the *Frenkel-Reshetikhin equations* (or *qKZ equations*)

$$\psi(pz_1, pz_2, \ldots, pz_i, z_{i+1}, \ldots, z_N) = \zeta R^2_{V_1(z_1)\otimes\cdots\otimes V_i(z_i), V_{i+1}(z_{i+1})\otimes\cdots\otimes V_N(z_N)} \psi,$$

where $R_{V_1(z_1),V_2(z_2)}$ is the R-matrix for affine algebras. There is a dual equation $\psi(p\zeta) = |R^2|\psi$, and the interaction of the two equations leads to *3D mirror symmetry* (or *symplectic duality*, see the review paper by Kamnitzer), which then leads to the *q-Langlands correspondence* (see the paper by Aganagic-Frenkel-Okounkov).