

MATH 7337: Harmonic Analysis

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Lecture 1

Aug. 19 — The Fourier Transform

1.1 The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over $\mathbb{R}$ unless otherwise specified.

Definition 1.1. The *Fourier transform* of $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi x} dx, \quad \xi \in \mathbb{R}.$$

Remark. Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \leq \int |f(x) e^{-2\pi i \xi x}| dx = \int |f(x)| dx = \|f\|_1 < \infty,$$

so $\widehat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$ (in fact, $\widehat{f}$ is continuous).

Remark. The Fourier transform is an operator $\mathcal{F} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ as $\|\widehat{f}\|_\infty = \text{ess sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq \|f\|_1$. This is linear in f . The *operator norm* of $\mathcal{F}$ is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1 \rightarrow L^\infty} = \sup_{\|f\|_1=1} \|\widehat{f}\|_\infty \leq \sup_{\|f\|_1=1} \|f\|_1 = 1,$$

so $\mathcal{F}$ is a bounded linear operator. However, $\mathcal{F}$ is not isometric (norm-preserving) in general.

Remark. Observe that

$$\widehat{f}(0) = \int f(x) e^{-2\pi i \cdot 0 \cdot x} dx = \int f(x) dx.$$

So if $f \geq 0$ and we normalize f so that $\widehat{f}(0) = 1$, then we have

$$|\widehat{f}(\xi)| \leq \int f(x) dx = \widehat{f}(0),$$

and so $\|\widehat{f}\|_\infty = \text{ess sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq 1$. This is one particular case where $\mathcal{F}$ does preserve the norm.

Definition 1.2. For $r \neq 0$, *dilation* of f by r is $f_r(x) = r f(rx)$. Note that $\|f_r\|_1 = \|f\|_1$.

Example 1.2.1. The *Dirichlet function* is $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$.¹ Note that $d \notin L^1(\mathbb{R})$. We can also define the *sinc* function as $\text{sinc } \xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$.

¹Recall that $C_0(\mathbb{R})$ is the space of continuous functions $f : \mathbb{R} \rightarrow \mathbb{C}$ such that $\lim_{x \rightarrow \pm\infty} f(x) = 0$.

However, d is the Fourier transform of a function in $L^1(\mathbb{R})$. Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \leq T, \\ 0 & |x| > T. \end{cases}$$

Note that $\chi_{-[T,T]} \in L^1(\mathbb{R})$. Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^T e^{-2\pi i \xi x} dx = \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \Big|_{-T}^T = \frac{\sin(2\pi T \xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$.

Remark. We will see in general that $\mathcal{F} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$, this is the Riemann-Lebesgue lemma. The image of $\mathcal{F}$ is a proper dense subspace of $C_0(\mathbb{R})$, which implies that $\mathcal{F}^{-1}$ must be unbounded as a linear operator by Banach space theory.

Proposition 1.1. *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}$, i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \rightarrow 0} 0,$$

where $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$.

Proof. We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}) dx \right| \leq \int |f(x)| |e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| dx.$$

Note that $f \in L^1(\mathbb{R})$ and $|e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| = |1 - e^{2\pi i \eta x}| \rightarrow 0$ as $\eta \rightarrow 0$ independent of ξ , so the statement follows from the dominated convergence theorem (the integrand is dominated by $2|f|$). $\square$

1.2 Motivation for the Fourier Transform

Remark. We will define the *inverse Fourier transform* of $f \in L^1(\mathbb{R})$ as

$$\check{f}(x) = \int f(\xi) e^{2\pi i \xi x} d\xi.$$

Note that $\check{f}(\xi) = \widehat{f}(-\xi)$. With enough assumptions, this is an inverse to the Fourier transform.

Proposition 1.2 (Fourier inversion formula). *If $f, \widehat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

Remark. Note that $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$ and $e_\xi : \mathbb{R} \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We have $e_\xi(x + y) = e_\xi(x) e_\xi(y)$, so e_ξ is a homomorphism, and it is also continuous. Thus e_ξ is a *character* on $\mathbb{R}$ (in fact, every character on $\mathbb{R}$ is of the form e_ξ for some ξ). One can use this idea to define Fourier transforms in much more general settings.

Remark. The Fourier transform decomposes a function f into the pure harmonics e_ξ , and the inversion formula says that we can recover f as a “sum” of these pure harmonics.

Lecture 2

Aug. 21 — The Riemann-Lebesgue Lemma

2.1 Properties of the Fourier Transform

Definition 2.1. Define the following operators:

1. *Translation:* $T_a f(x) = f(x - a)$ for $a \in \mathbb{R}$;
2. *Modulation:* $M_b f(x) = e^{2\pi i b x} f(x)$ for $b \in \mathbb{R}$;
3. *Dilation:* $f_\lambda(x) = \lambda f(\lambda x)$ for $\lambda > 0$;
4. *Involution:* $\tilde{f}(x) = \overline{f(-x)}$.

Remark. Translation and modulation are isometries on $L^p(\mathbb{R})$ for any p . Dilation as defined above is L^1 -normalized, so it is only an isometry on $L^1(\mathbb{R})$.

Exercise 2.1. If $f \in L^1(\mathbb{R})$, then

1. $(T_a f)^\wedge(\xi) = (M_{-a} \hat{f})(\xi) = e^{-2\pi i \xi a} \hat{f}(\xi)$;
2. $(M_b f)^\wedge(\xi) = (T_b \hat{f})(\xi) = \hat{f}(\xi - b)$;
3. $(f_\lambda)^\wedge(\xi) = \lambda (f_{1/\lambda})^\wedge(\xi) = \hat{f}(\xi/\lambda)$;¹
4. $(\bar{f})^\wedge(\xi) = (\hat{f})^\sim(\xi) = \overline{\hat{f}(-\xi)}$;
5. $(\tilde{f})^\wedge(\xi) = \overline{\hat{f}(\xi)}$.

2.2 The Riemann-Lebesgue Lemma

Definition 2.2. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support. For a continuous function, the *support* of f , denoted $\text{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. So for a continuous function f , $\text{supp}(f)$ is compact if and only if $f = 0$ outside some finite interval.

Theorem 2.1. $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. In other words,

1. the closure of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$ is all of $L^p(\mathbb{R})$;
2. for any $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, there exists $g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$;
3. if $f \in L^p(\mathbb{R})$, then there exists $g_n \in C_c(\mathbb{R})$ such that $g_n \rightarrow f$ in L^p -norm, i.e. $\|g_n - f\|_p \rightarrow 0$.

¹Note that the result is an L^∞ -normalized dilation.

For $p = \infty$, $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ with respect to the L^∞ -norm (this is the same as the uniform norm for continuous functions).

Proof. We sketch the proof. First approximate $f \in L^p(\mathbb{R})$ by a simple function (one that takes only finitely many distinct values) $\phi = \sum_{k=1}^N c_k \chi_{E_k}$, e.g. by rounding down to the nearest integer multiple of 2^{-n} . Then use Urysohn's lemma to approximate χ_{E_k} by a continuous function. $\square$

Exercise 2.2. Fix $1 \leq p < \infty$. Prove that if $f \in L^p(\mathbb{R})$, then $\lim_{a \rightarrow 0} \|f - T_a f\|_p = 0$. We say that translation is *strongly continuous* on $L^p(\mathbb{R})$. For $p = \infty$, use $C_0(\mathbb{R})$ and the uniform norm instead.

Lemma 2.1 (Riemann-Lebesgue lemma). *If $f \in L^1(\mathbb{R})$, then $\widehat{f} \in C_0(\mathbb{R})$,*

Proof. We have already seen that $\widehat{f}$ is continuous. So it suffices to show decay at ∞ . Write

$$\widehat{f}(\xi) = - \int f(x) e^{-2\pi i \xi x} e^{-2\pi i \xi (1/2\xi)} dx = - \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi (x+1/2\xi)} dx.$$

Now make the change of variables $x \mapsto x - 1/2\xi$, so we get

$$\widehat{f}(\xi) = - \int_{-\infty}^{\infty} f\left(x - \frac{1}{2\xi}\right) e^{-2\pi i \xi x} dx = - \int T_{1/2\xi} f(x) e^{-2\pi i \xi x} dx.$$

Taking an average with the usual expression for $\widehat{f}(\xi)$, we have

$$\widehat{f}(\xi) = \frac{1}{2} \int (f(x) - T_{1/2\xi} f(x)) e^{-2\pi i \xi x} dx.$$

Taking absolute values, we obtain

$$|\widehat{f}(\xi)| \leq \frac{1}{2} \int |f(x) - T_{1/2\xi} f(x)| dx = \frac{1}{2} \|f - T_{1/2\xi} f\|_1 \xrightarrow{\xi \rightarrow \pm\infty} 0$$

by the strong continuity of translation on $L^1(\mathbb{R})$. $\square$

Exercise 2.3. The following is an alternative proof of the Riemann-Lebesgue lemma. Recall that we have $\widehat{\chi}_{-T,T} = d_{2\pi T} \in C_0(\mathbb{R})$. By taking translations and dilations, we see that $\widehat{\chi}_{[a,b]} \in C_0(\mathbb{R})$. Consider *really simple functions* $\phi = \sum_{k=1}^N c_k \chi_{[a_k, b_k]}$, and by linearity we can write

$$\widehat{\phi} = \sum_{k=1}^N c_k \widehat{\chi}_{[a_k, b_k]} \in C_0(\mathbb{R}).$$

Note that really simple functions are also dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. So if $f \in L^1(\mathbb{R})$, there exist really simple $\phi_n \rightarrow f$ in L^1 -norm. On the Fourier side, we have

$$\|\widehat{f} - \widehat{\phi}_n\|_\infty \leq \|f - \phi_n\|_1 \rightarrow 0.$$

Since $\phi_n \rightarrow f$ uniformly and $C_0(\mathbb{R})$ is a Banach space, we conclude $\widehat{f} \in C_0(\mathbb{R})$. Fill in the details.

2.3 Position and Momentum Operators

Definition 2.3. The *position operator* $P : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ is given by $Pf(x) = xf(x)$. Note that P is unbounded on $L^1(\mathbb{R})$ (in fact, P is not defined on all of $L^1(\mathbb{R})$). Restrict P to the domain

$$D_P = \{f \in L^1(\mathbb{R}) : xf(x) \in L^1(\mathbb{R})\},$$

which is dense in $L^1(\mathbb{R})$. Note that D_P cannot be bounded as it does not admit an extension to $L^1(\mathbb{R})$.

Exercise 2.4. Show that $\sup_{\|f\|_1=1, f \in D_P} \|Pf\|_1 = \infty$.

Definition 2.4. The *momentum operator* $M : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ is given by $Mf = f'/2\pi i$. Similarly, M is unbounded and defined only on a dense subset of $L^1(\mathbb{R})$.

Remark. We have the relation $(Mf)^\wedge(\xi) = \xi P\hat{f}(\xi)$, whenever the statement makes sense.

2.4 The HRT Conjecture

Conjecture 2.1 (HRT conjecture). Assume g is not zero a.e., a_k, b_k are distinct, and consider finite linear combinations of translations and modulations of $g \in L^2(\mathbb{R})$ of the following form:

$$\sum_{k=1}^N c_k e^{2\pi i b_k x} g(x - a_k). \quad (*)$$

If $(*) = 0$, then must it be that $c_1 = \dots = c_N = 0$? In other words, are these linearly independent?

Remark. Consider the special case $b_k = 0$ for every k , so $\sum c_k T_{a_k} g = \sum c_k g(x - a_k) = 0$ a.e. Then

$$\left(\sum c_k T_{a_k} g \right)^\wedge = \sum c_k M_{-a_k} \hat{g} = \left(\sum_{k=1}^N c_k e^{-2\pi i a_k \xi} \right) \hat{g}(\xi) = 0.$$

Since $\hat{g}$ is not zero a.e., we must have $\sum_{k=1}^N c_k e^{-2\pi i a_k \xi} = 0$, which implies $c_k = 0$ for all k . In particular, this means that translations alone are linearly independent (the same is true for modulations alone).

Remark. The general case of the HRT conjecture is still open. Note that after taking a Fourier transform, we end up with the same problem, just for $\hat{g}$ instead of g .

Lecture 3

Aug. 3 — Convolution

3.1 Convolution

Definition 3.1. If f, g are measurable on $\mathbb{R}$, their *convolution* is (formally)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y) dy.$$

Remark. When it exists, we have

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y) dy = \int_{-\infty}^{\infty} f(x - y)g(y) dy = (g * f)(x)$$

by the change of variables $y \mapsto x - y$. So $f * g = g * f$, if it exists. Similarly, $f * (g * h) = (f * g) * h$ if each of these convolutions exist.

Remark. If we take $g_T = \chi_{-T,T}/2T$ (note that $\|g_T\|_1 = 1$), then

$$(f * g_T)(x) = \int_{-\infty}^{\infty} f(y)g_T(x - y) dy = \frac{1}{2T} \int_{x-T}^{x+T} f(y) dy = \text{Avg}_{[-T,T]} f(x),$$

so we can see convolution as a averaging or smoothing operation (also known as *mollification*).

Remark. We would like to show $f, g \in L^1(\mathbb{R})$ implies $f * g \in L^1(\mathbb{R})$. Note that $(f * g)^\wedge = \widehat{f\widehat{g}} \in C_0(\mathbb{R})$, since $C_0(\mathbb{R})$ is closed under multiplication, even though $L^1(\mathbb{R})$ is not.

Remark. The *Lebesgue differentiation theorem* says that if $f \in L^1_{\text{loc}}(\mathbb{R})$, then $(f * g_T)(x) \rightarrow f(x)$ a.e.

3.2 Properties of Convolution

Remark. Use the notation

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx,$$

whenever this integral exists. Then *Hölder's inequality* says that if $1/p + 1/p' = 1$ with $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$ and we have

$$|\langle f, g \rangle| \leq \int |f(x)||g(x)| dx \leq \|f\|_p \|g\|_{p'}.$$

Theorem 3.1. For $1 \leq p \leq \infty$, if $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^\infty(\mathbb{R})$.

Proof. By Hölder's inequality, we can write

$$\int |f(y)g(x-y)| dy \leq \|f\|_p \|g(x)\|_{p'} < \infty,$$

so $(f * g)(x)$ exists for every $x \in \mathbb{R}$. □

Exercise 3.1. Show that $f * g \in C_b(\mathbb{R}) = \{h : \mathbb{R} \rightarrow \mathbb{C} : h \text{ is continuous and bounded}\}$.

Remark. Denote $g^*(y) = \overline{g(-y)}$. Then we have

$$(f * g)(x) = \int f(y)g(x-y) dy = \int f(y)\overline{g^*(y-x)} dy = \langle f, T_x g^* \rangle.$$

Theorem 3.2. Let $f, g \in L^1(\mathbb{R})$. Then

1. $f(y)g(x-y)$ is measurable and integrable on $\mathbb{R}^2$;
2. for a.e. $x \in \mathbb{R}$, $f(y)g(x-y)$ is measurable and integrable on $\mathbb{R}$ as a function of y ;
3. $f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, i.e. convolution is submultiplicative on $L^1(\mathbb{R})$;
4. $(f * g)^\wedge(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for every $\xi \in \mathbb{R}$.

Proof. (1) Let $h(x, y) = f(y)g(x-y)$. Then we have

$$\{h > a\} = h^{-1}((a, \infty)) = \{(x, y) : f(y)g(x-y) > a\} = \{f > a\} \times \mathbb{R},$$

which is measurable in $\mathbb{R}^2$ since $\{f > a\}$ and $\mathbb{R}$ are measurable in $\mathbb{R}$. Similarly, $g(y)$ is measurable on $\mathbb{R}^2$, so $F(x, y) = f(y)g(x-y)$ is measurable on $\mathbb{R}^2$. Now make a linear change of variables $T(x, y) = (y, x-y)$, so $H = F \circ T = f(y)g(x-y)$ is measurable (note that linear maps preserve measurability).

Now we can integrate by Tonelli's theorem and see that

$$\begin{aligned} \iint |f(y)g(x-y)| dx dy &= \int |f(y)| \left(\int |g(x-y)| dx \right) dy = \int |f(y)| \left(\int |g(z)| dz \right) dy \\ &= \int |f(y)| \|g\|_1 dy = \|f\|_1 \|g\|_1 < \infty, \end{aligned}$$

hence $f(y)g(x-y)$ is integrable on $\mathbb{R}^2$.

(2) This follows by Fubini's theorem since $f(y)g(x-y)$ is integrable.

(3) By (2), $(f * g)(x)$ exists for a.e. x , and

$$\int |(f * g)(x)| dx = \int \left| \int f(y)g(x-y) dy \right| dx \leq \iint |f(y)g(x-y)| dy dx \leq \|f\|_1 \|g\|_1,$$

which is the desired inequality.

(4) Justify the following calculation as an exercise via Fubini/Tonelli's theorem:

$$\begin{aligned}(f * g)^\wedge(\xi) &= \int (f * g)(x) e^{-2\pi i \xi x} dx = \int \left(\int f(y) g(x - y) dy \right) e^{-2\pi i \xi x} dx \\ &= \iint f(y) e^{-2\pi i \xi y} g(x - y) e^{-2\pi i \xi (x - y)} dy dx.\end{aligned}$$

By Fubini's theorem, we can exchange orders and write

$$\begin{aligned}(f * g)^\wedge(\xi) &= \int f(y) e^{-2\pi i \xi y} \left(\int g(x - y) e^{-2\pi i \xi (x - y)} dx \right) dy \\ &= \int f(y) e^{-2\pi i \xi y} \left(\int g(z) e^{-2\pi i \xi z} dz \right) dy = \widehat{f}(\xi) \widehat{g}(\xi),\end{aligned}$$

which is the desired equality. □

Corollary 3.2.1. $L^1(\mathbb{R})$ is closed under convolution.

Definition 3.2. An *algebra* is a vector space A with a product such that

- (a) $(fg)h = f(gh)$,
- (b) $f(g + h) = fg + fh$,
- (c) $\alpha(fg) = (\alpha f)g = f(\alpha g)$.

If $fg = gf$ always, then we say that A is *commutative*. A Banach space which is also an algebra with a submultiplicative product is a *Banach algebra*.

Example 3.2.1. With convolution as a product, $L^1(\mathbb{R})$ becomes a commutative Banach algebra without identity. Similarly, $C_0(\mathbb{R})$ is also a commutative Banach algebra without identity (under pointwise products). The space $\mathcal{B}(X)$ of bounded linear operators on a Banach space X is also a Banach space under the operator norm, and we have $\|AB\| \leq \|A\|\|B\|$ with composition as a product. So $\mathcal{B}(X)$ is a noncommutative Banach algebra, with identity.

3.3 Young's Inequality

Theorem 3.3 (Young's inequality, special case). Fix $1 \leq p \leq \infty$. If $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^p(\mathbb{R})$ and $\|f * g\|_p \leq \|f\|_p \|g\|_1$.

Proof. The case $p = \infty$ is easy by Hölder's inequality and $p = 1$ is done, so assume $1 < p < \infty$. Then

$$|(f * g)(x)| \leq \int |f(y)| |g(x - y)| dy = \int (|f(y)| |g(x - y)|^{1/p}) (|g(x - y)|^{1/p'}) dy,$$

By Hölder's inequality, we can write

$$\begin{aligned}|(f * g)(x)| &\leq \left(\int |f(y)|^p |g(x - y)| dy \right)^{1/p} \left(\int |g(x - y)| dy \right)^{1/p'} \\ &\leq \|g\|_1^{1/p'} \left(\int |f(y)|^p |g(x - y)| dy \right)^{1/p}.\end{aligned}$$

Now taking L^p -norms, we get

$$\|f * g\|_p^p = \int |(f * g)(x)|^p dx \leq \|g\|_1^{p/p'} \iint |f(y)|^p |g(x - y)| dy dx.$$

By Tonelli's theorem, we can exchange orders and write

$$\|f * g\|_p^p \leq \|g\|_1^{p/p'} \int |f(y)|^p \left(\int |g(x - y)| dx \right) dy \leq \|g\|_1^{1+p/p'} \|f\|_p^p = \|g\|_1^p \|f\|_p^p,$$

so we get the desired inequality $\|f * g\|_p \leq \|f\|_p \|g\|_1$ after taking p th roots. $\square$

Exercise 3.2 (Young's inequality, general case). Let $1 \leq p, q, r \leq \infty$ satisfy $1/r = 1/p + 1/q - 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Remark. Recall *Minkowski's inequality* (the triangle inequality in $L^p(\mathbb{R})$):

$$\left\| \sum f_k \right\|_p \leq \sum \|f_k\|_p.$$

Minkowski's integral inequality then says that for $1 \leq p \leq \infty$,

$$\left\| \int f_x dx \right\|_p = \left(\int \left| \int f(x, y) dx \right|^p dy \right)^{1/p} \leq \int \left(\int |f(x, y)|^p dy \right)^{1/p} dx = \int \|f_x\|_p dx.$$

One can also use this to prove Young's inequality.

Remark. The *Babenko-Beckner constant* is the optimal constant in front of Hölder's inequality:

$$A_p = \left(\frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}.$$

The optimal constant in Young's inequality is $A_p A_q A_{r'}$, i.e. we have

$$\|f * g\|_r \leq (A_p A_q A_{r'}) \|f\|_p \|g\|_q.$$

3.4 The Dirac Delta

Remark. Is there an identity for convolution? Suppose there was a function $\delta \in L^1(\mathbb{R})$ (the *Dirac delta function*) such that $f * \delta = f$ for all $f \in L^1(\mathbb{R})$. Then we have $(f * \delta)^\wedge = \widehat{f}$, so

$$\widehat{f}(\xi) \widehat{\delta}(\xi) = \widehat{f}(\xi) \quad \text{for all } f \in L^1(\mathbb{R}).$$

Take $f(x) = e^{-x^2}$ with $\widehat{f}(\xi) = e^{-\xi^2}$ and note that $\widehat{f}(\xi)$ is everywhere nonzero. Then $\widehat{\delta}(\xi) = 1$ for all $\xi \in \mathbb{R}$, which contradicts the Riemann-Lebesgue lemma.

The correct way to work with the Dirac delta is to use the measure

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

One can then integrate against the measure δ to achieve a similar effect.

Lecture 4

Aug. 28 — Convolution, Part 2

Lecture 5

Sept. 2 — Smoothness and Decay

5.1 Smoothness and Decay

Theorem 5.1 (Decay in time implies smoothness in frequency). Assume $f \in L^1(\mathbb{R})$ and $x^m f(x) \in L^1(\mathbb{R})$, where $m > 0$. Then

$$\widehat{f} \in C_0^m(\mathbb{R}) = \{g : g, g', \dots, g^{(m)} \in C_0(\mathbb{R})\}.$$

Furthermore, we have

$$\widehat{f}^{(k)} = \frac{d^k}{d\xi^k} \widehat{f} = ((-2\pi i x)^k f(x))^\wedge.$$

Proof. The proof is by induction on m . When $m = 1$, we can formally write

$$\begin{aligned} \frac{d}{d\xi} \widehat{f}(\xi) &= \frac{d}{d\xi} \int f(x) e^{-2\pi i \xi x} dx \\ &\stackrel{(*)}{=} \int f(x) \frac{d}{d\xi} e^{-2\pi i \xi x} dx = \int f(x) (-2\pi i x) e^{-2\pi i \xi x} dx = (-2\pi i x f(x))^\wedge(\xi). \end{aligned}$$

It suffices to justify step (*), which we will do by appealing to the dominated convergence theorem. We can write

$$\widehat{f}'(\xi) = \lim_{\eta \rightarrow 0} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \rightarrow 0} \int f(x) \frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} dx.$$

Note that we have the pointwise limit

$$f(x) \frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} \xrightarrow{\eta \rightarrow 0} f(x) \frac{d}{d\xi} e^{-2\pi i \xi x} = -2\pi i x f(x) e^{-2\pi i \xi x}.$$

Also note that we can bound

$$\left| f(x) \frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} \right| = \left| f(x) \frac{e^{-2\pi i \eta x} - 1}{\eta} \right| \leq \left| f(x) \frac{-2\pi i \eta x}{\eta} \right| = |2\pi x f(x)|,$$

where we noted that $|e^{i\theta} - 1| \leq |\theta|$ for $\theta \in \mathbb{R}$. Thus $2\pi x f(x)$ dominates the integrand and is integrable since $x f(x) \in L^1(\mathbb{R})$ by assumption, we can conclude (*) by the dominated convergence theorem. Then $\widehat{f}' \in C_0(\mathbb{R})$ by the Riemann-Lebesgue lemma, since $\widehat{f}' = (-2\pi i x f(x))^\wedge$ where $-2\pi i x f(x) \in L^1(\mathbb{R})$.

The inductive step is part of Homework 1. □

Remark. Recall the position and momentum operators $Pf(x) = xf(x)$ and $Mf(x) = f'(x)/2\pi i$. If $f, Pf \in L^1(\mathbb{R})$, then the above theorem tells us that $(Pf)^\wedge = -M\widehat{f}$.

5.2 Absolute Continuity

Definition 5.1. A function $f : [a, b] \rightarrow \mathbb{C}$ is *absolutely continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\{[a_j, b_j]\}_j$ are countably many non-overlapping intervals, then

$$\sum_j (b_j - a_j) < \delta \quad \text{implies} \quad \sum_j |f(b_j) - f(a_j)| < \epsilon.$$

Define $\text{AC}_{\text{loc}}(\mathbb{R}) = \{f \in C(\mathbb{R}) : f \text{ is absolutely continuous on every interval } [a, b]\}$.

Theorem 5.2 (Fundamental theorem of calculus). *If $g : [a, b] \rightarrow \mathbb{C}$, then the following are equivalent:*

1. $g \in \text{AC}[a, b]$;
2. there exists $f \in L^1[a, b]$ such that for all $x \in [a, b]$,

$$g(x) - g(a) = \int_a^x f(t) dt;$$

3. g is differentiable at a.e. point, $g' \in L^1[a, b]$, and

$$g(x) - g(a) = \int_a^x g'(t) dt.$$

Remark. The Cantor-Lebesgue function $\varphi : [0, 1] \rightarrow [0, 1]$ is continuous with $\varphi' = 0$ a.e., but

$$\int_0^1 \varphi'(x) dx = 0 \neq 1 = \varphi(1) - \varphi(0).$$

Lemma 5.1 (Growth lemma). *If $f : [a, b] \rightarrow \mathbb{R}$ is measurable and differentiable at every point in a measurable set $E \subseteq [a, b]$, then*

$$|f(E)|_e \leq \int_E |f'|,$$

where $|f(E)|_e$ denotes the exterior Lebesgue measure of $f(E)$.

Theorem 5.3 (Banach-Zaretsky theorem). *If $f : [a, b] \rightarrow \mathbb{R}$, then the following are equivalent:*

1. $f \in \text{AC}[a, b]$;
2. f is continuous, f has bounded variation, and $|A| = 0$ implies $|f(A)| = 0$;
3. f is continuous and differentiable a.e., $f' \in L^1[a, b]$, and $|A| = 0$ implies $|f(A)| = 0$.

Theorem 5.4. *If $f : [a, b] \rightarrow \mathbb{C}$ is differentiable on $[a, b]$ and $f' \in L^1[a, b]$, then $f \in \text{AC}[a, b]$.*

Proof. By the Banach-Zaretsky theorem, it suffices to show that $|A| = 0$ implies $|f(A)| = 0$. If $|A| = 0$, then by the growth lemma,

$$|f(A)| \leq \int_A |f'| = 0,$$

which completes the proof. (Technically we should split f into its real and imaginary parts.) □

5.3 Smoothness and Decay, Continued

Theorem 5.5 (Smoothness in time implies decay in frequency). *If $f \in L^1(\mathbb{R})$ is everywhere m -times differentiable and $f, f', \dots, f^{(m)} \in L^1(\mathbb{R})$, then*

$$\widehat{f^{(k)}}(\xi) = (2\pi i \xi)^k \widehat{f}(\xi), \quad \text{for } k = 0, \dots, m,$$

hence $|\widehat{f}(\xi)| \leq |2\pi \xi|^{-k} |\widehat{f^{(k)}}(\xi)| \leq |2\pi \xi|^{-k} \|\widehat{f^{(k)}}\|_\infty \leq |2\pi \xi|^{-k} \|f^{(k)}\|_1$ for $k = 0, \dots, m$.

Proof. We prove only the case $m = 1$, the rest follows by induction. Assume $f, f' \in L^1(\mathbb{R})$. By Theorem 5.4, we have $f \in \text{AC}_{\text{loc}}(\mathbb{R})$. Hence by the fundamental theorem of calculus,

$$f(x) - f(0) = \int_0^x f'(t) dt.$$

Because f' is integrable, we get that

$$\lim_{x \rightarrow \infty} f(x) = f(0) + \lim_{x \rightarrow \infty} \int_0^x f'(t) dt = f(0) + \int_0^\infty f'(t) dt.$$

Since f is integrable and this limit exists, the limit must be 0. Hence $f \in C_0(\mathbb{R})$. We can compute

$$\widehat{f'}(\xi) = \int_{-\infty}^\infty f'(x) e^{-2\pi i \xi x} dx = \lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \int_a^b f'(x) e^{-2\pi i \xi x} dx.$$

Since f is absolutely continuous, we can integrate by parts to get

$$\widehat{f'}(\xi) = \lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \left[f(b) e^{-2\pi i \xi b} - f(a) e^{-2\pi i \xi a} + (2\pi i \xi) \int_a^b f(x) e^{-2\pi i \xi x} dx \right] = (2\pi i \xi) \widehat{f}(\xi),$$

which proves the desired result. □

Remark. Note that for the absolute continuity arguments, we need to first restrict to a finite interval and then take limits, since we only know that $f \in \text{AC}_{\text{loc}}(\mathbb{R})$.

5.4 Approximate Identities

Remark. Recall that if we take $g_T = \chi_{[-T, T]}/2T$, then we have $(f * g_T)(x) = \text{Avg}_{[x-T, x+T]} f$. As $T \rightarrow 0$, this converges to f if f is continuous, and converges a.e. to f if f is integrable. In particular, this is almost like a identity for the convolution operation.

Definition 5.2. If $k_\lambda \in L^1(\mathbb{R})$ for $\lambda > 0$ (or sometimes $\lambda \in \mathbb{N}$) satisfy:

- (a) Normalization: $\int_{-\infty}^\infty k_\lambda = 1$ for every λ ,
- (b) L^1 -boundedness: $\sup_\lambda \|k_\lambda\|_1 = \sup_\lambda \int_{-\infty}^\infty |k_\lambda| < \infty$,
- (c) L^1 -concentration: $\lim_{\lambda \rightarrow \infty} \int_{|x| \geq \delta} |k_\lambda| = 0$ for every $\delta > 0$,

then we say that $\{k_\lambda\}$ is an *approximate identity (for convolution)*.

Exercise 5.1. If $k \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} k = 1$, then $k_\lambda(x) = \lambda k(\lambda x)$ forms an approximate identity.

Remark. If we choose k_λ to be nice, then $f * k_\lambda$ will also be nice and “close” to f .

Lecture 6

Sept. 4 — Approximate Identities

6.1 Properties of Approximate Identities

Theorem 6.1. *If $\{k_\lambda\}$ is an approximate identity, then for all $f \in L^1(\mathbb{R})$,*

$$\lim_{\lambda \rightarrow \infty} \|f * k_\lambda - f\|_1 = 0.$$

*That is, $f * k_\lambda \rightarrow f$ in L^1 -norm.*

Proof. We have already seen that $f * k_\lambda \in L^1(\mathbb{R})$. Then

$$\|f - f * k_\lambda\|_1 = \int |f(x) - (f * k_\lambda)(x)| dx = \int \left| f(x) \int k_\lambda(t) dt - \int f(x-t) k_\lambda(t) dt \right| dx,$$

where we used that $\int k_\lambda(t) dt = 1$. Collecting terms and taking absolute values inside,

$$\|f - f * k_\lambda\|_1 \leq \iint |f(x) - f(x-t)| |k_\lambda(t)| dt dx.$$

By Tonelli's theorem, we can exchange orders to get

$$\|f - f * k_\lambda\|_1 \leq \int |k_\lambda(t)| \left(\int |f(x) - T_t f(x)| dx \right) dt = \int |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

We split this integral into two parts:

$$\|f - f * k_\lambda\|_1 \leq \int_{|t| < \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt + \int_{|t| \geq \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

By the strong continuity of translation, we know that $\lim_{t \rightarrow 0} \|f - T_t f\|_1 = 0$, so for any $\epsilon > 0$, there exists $\delta > 0$ such that $|t| < \delta$ implies $\|f - T_t f\|_1 < \epsilon$. This lets us estimate the first integral:

$$\|f - f * k_\lambda\|_1 \leq \epsilon \int_{|t| < \delta} |k_\lambda(t)| dt + \int_{|t| \geq \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

For the second integral, we can use $\|f - T_t f\|_1 \leq \|f\|_1 + \|T_t f\|_1 = 2\|f\|_1$ to get

$$\|f - f * k_\lambda\|_1 \leq \epsilon \int_{|t| < \delta} |k_\lambda(t)| dt + 2\|f\|_1 \int_{|t| \geq \delta} |k_\lambda(t)| dt \leq \epsilon K + 2\|f\|_1 \epsilon$$

where $K = \sup_\lambda \|k_\lambda\|_1 < \infty$ and λ is large enough (as $\int_{|t| \geq \delta} |k_\lambda(t)| dt \rightarrow 0$). So $\|f - f * k_\lambda\|_1 \rightarrow 0$. $\square$

Exercise 6.1. Show that for $1 \leq p < \infty$, we still have $\|f - f * k_\lambda\|_p \rightarrow 0$ as $\lambda \rightarrow \infty$ for $f \in L^p(\mathbb{R})$. For $p = \infty$, show that if $f \in C_0(\mathbb{R})$, then $\|f - f * k_\lambda\|_\infty \rightarrow 0$ as $\lambda \rightarrow \infty$, that is $f * k_\lambda \rightarrow f$ uniformly.

Exercise 6.2. Show that if $f \in C_b(\mathbb{R})$, then for every compact set $K \subseteq \mathbb{R}$,

$$\lim_{\lambda \rightarrow \infty} \|(f - f * k_\lambda)\chi_K\|_\infty = 0.$$

Definition 6.1. A function f is *Hölder continuous* with exponent $\alpha > 0$ if

$$|f(x) - f(y)| \leq K|x - y|^\alpha$$

for some constant K and all x, y . If $\alpha = 1$, then we say that f is *Lipschitz*.

Remark. If f is Hölder continuous with exponent $\alpha > 1$, then the mean value theorem implies that f is constant. Thus the interesting range for Hölder continuity is $0 < \alpha \leq 1$.

Exercise 6.3. Let f be bounded and Hölder continuous with exponent $0 < \alpha \leq 1$, then show that

$$f * k_\lambda \rightarrow f \quad \text{uniformly on } \mathbb{R}.$$

Remark. If f is differentiable and f' is bounded, then f is Lipschitz.

Remark. Recall the *Lebesgue differentiation theorem*, which says that if $f \in L^1_{\text{loc}}(\mathbb{R})$, then

$$(f * g_T)(x) = \frac{1}{2T} \int_{x-T}^{x+T} f(t) dt \longrightarrow f(x) \quad \text{for a.e. } x.$$

where $g_T = \chi_{[-T, T]}/(2T)$. The points where the limit holds are called the *Lebesgue points* of f .

Theorem 6.2. Assume k is bounded and compactly supported and $\int k = 1$. Set $k_\lambda(x) = \lambda k(\lambda x)$ for $\lambda > 0$. Then for any $f \in L^1(\mathbb{R})$,

$$f * k_\lambda \rightarrow f \quad \text{pointwise a.e.}$$

Moreover, the pointwise limit holds at every Lebesgue point of f .

Proof. Assume $\text{supp}(k) \subseteq [-R, R]$. We can write

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} |f(x) - (f * k_\lambda)(x)| &= \lim_{\lambda \rightarrow \infty} \left| f(x) \int k_\lambda(x - t) dt - \int f(x) k_\lambda(x - t) dt \right| \\ &\leq \lim_{\lambda \rightarrow \infty} \int |f(x) - f(t)| \lambda |k(\lambda x - \lambda t)| dt \\ &\leq \lim_{\lambda \rightarrow \infty} \lambda \int_{x-R/\lambda}^{x+R/\lambda} |f(x) - f(t)| |k(\lambda x - \lambda t)| dt. \end{aligned}$$

Making a change of variables $T = R/\lambda$, we have

$$\lim_{\lambda \rightarrow \infty} |f(x) - (f * k_\lambda)(x)| \leq \lim_{T \rightarrow 0} \frac{1}{2T} \int_{x-T}^{x+T} |f(x) - f(t)| dt \cdot \|k\|_\infty = 0$$

for every Lebesgue point x by the Lebesgue differentiation theorem. □

6.2 Density Results and Smooth Urysohn Lemma

Theorem 6.3. $C_c^m(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $m > 0$ and $1 \leq p < \infty$.

Proof. Fix $\epsilon > 0$. Choose $k \in C_c^m(\mathbb{R})$ with $\int k = 1$, and set $k_\lambda(x) = \lambda k(\lambda x)$. Note that there exists a compactly supported $g \in L^p(\mathbb{R})$ with $\|f - g\|_p < \epsilon$ (e.g. take $g = f\chi_{[-R,R]}$ for large enough R , this works since $f\chi_{[-R,R]}$ converges pointwise to f as $R \rightarrow \infty$ and is dominated by f , so the dominated convergence theorem implies that $f\chi_{[-R,R]} \rightarrow f$ in L^p -norm). Then note that $g * k_\lambda \in C_c^m(\mathbb{R})$ and $g * k_\lambda \rightarrow g$ in L^p -norm, so there exists λ such that $\|g - g * k_\lambda\|_p < \epsilon$. Thus

$$\|f - g * k_\lambda\|_p \leq \|f - g\|_p + \|g - g * k_\lambda\|_p < 2\epsilon$$

which implies the desired result. $\square$

Corollary 6.3.1. $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Remark. The above proof would work for $m = 0$ but becomes circular: The step $g * k_\lambda \in C_c^m(\mathbb{R})$ relies on the strong continuity of translation, which we proved by first showing it for $C_c(\mathbb{R})$ and then by an extension by density to $L^p(\mathbb{R})$. In particular, we needed to already know that $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Proposition 6.1 (C^∞ Urysohn's lemma). *If $K \subseteq \mathbb{R}$ is compact and $U \subseteq K$ is open, then there exists $f \in C_c^\infty(\mathbb{R})$ such that $0 \leq f \leq 1$, $f = 1$ on K , and $f = 0$ on U^c .*

Proof. Since K is compact and U^c is closed, we have

$$d = \text{dist}(K, U^c) = \inf\{|x - y| : x \in K, y \notin U\} > 0.$$

Set $V = \{y \in \mathbb{R} : \text{dist}(y, K) < d/3\}$, and choose any $k \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(k) \subseteq [-d/3, d/3]$ and $\int k = 1$. Take $f = k * \chi_V \in C_c^\infty(\mathbb{R})$, which has $\text{supp}(f) \subseteq \text{supp}(k) + V \subseteq U$. If $x \in K$, then

$$f(x) = \int_V k(x - y) dy = \int k = 1.$$

One can check that $0 \leq f \leq 1$ and $f = 0$ on U^c as an exercise, which would prove the result. $\square$

Lecture 7

Sept. 18 —

Lecture 8

Sept. 23 —

Lecture 9

Sept. 25 — Kernels and Schwartz Space

9.1 More Kernels

Remark. Recall that if $\{k_\lambda\}$ (where $k_\lambda(x) = \lambda k(\lambda x)$) is an approximate identity, then $f * k_\lambda \rightarrow f$ in L^p -norm. We also have

$$(f * k_\lambda)^\wedge = \widehat{f} \cdot \widehat{k}_\lambda.$$

Note that $\widehat{k}_\lambda(\xi) = \widehat{k}(\xi/\lambda)$, so $\widehat{k}_\lambda(\xi) \rightarrow 1$ pointwise. If the Dirichlet kernel $d_{2\pi\lambda}(\xi) = \sin(2\pi\lambda\xi)/(\pi\xi)$ were integrable, then we would have $\widehat{d_{2\pi\lambda}}(\xi) = \chi_{[-\lambda, \lambda]}(\xi)$, so we must have $d_{2\pi\lambda} \notin L^1(\mathbb{R})$.

Some alternatives are the following:

1. The Fejér kernel $w_\lambda(\xi) = d_{2\pi\lambda}^2(\xi) \in L^1(\mathbb{R})$. We saw this in the proof of the inversion formula.
2. The *de la Vallée Poussin kernel* v_λ , which has $\widehat{v}_\lambda$ as a trapezoid which is 1 on $[-\lambda, \lambda]$ and decays linearly to 0 on $[-2\lambda, -\lambda]$ and $[\lambda, 2\lambda]$. One has

$$(f * v_\lambda)^\wedge = \widehat{f} \cdot \widehat{v}_\lambda = \widehat{f} \quad \text{on } [-\lambda, \lambda].$$

Explicitly, one can define $v(x) = 2w_2(x) - w(x)$ and $v_\lambda(x) = \lambda v(\lambda x)$.

3. The *Poisson kernel* $p(x) = 1/\pi(x^2 + 1)$.
4. The *Gauss kernel* $\phi(x) = e^{-\pi x^2}$.

Exercise 9.1. Show that if $f \in L^1(\mathbb{R})$, then $\text{supp}(\widehat{f})$ is compact if and only if $f = f * g$ for some $g \in L^1(\mathbb{R})$. Hint: Use the de la Vallée Poussin kernel.

Exercise 9.2. Let $\Phi = \widehat{\phi}$, where ϕ is the Gauss kernel. Show that $\Phi'(\xi) = -2\pi\xi\Phi(\xi)$. Then solve this differential equation to get $\Phi(\xi) = \Phi(0)\phi(\xi)$. Finally show that $\Phi(0) = 1$, and conclude $\Phi(\xi) = \phi(\xi)$.

Remark. There are other ways to find functions which are their own Fourier transforms. The inversion formula says that if $f, \widehat{f} \in L^1(\mathbb{R})$, then we have

$$f(x) = (\widehat{f})^\vee(x) = f^{\wedge\wedge}(-x) = f^{\wedge\wedge\wedge\wedge}(x).$$

In particular, for f sufficiently nice, if we take $g = f + f^\wedge + f^{\wedge\wedge} + f^{\wedge\wedge\wedge}$, then $\widehat{g} = g$.

Theorem 9.1 (Weierstrass approximation theorem). *If $f \in C[a, b]$ and $\epsilon > 0$, then there exists a polynomial p such that $\|f - p\|_\infty = \sup_{x \in [a, b]} |f(x) - p(x)| < \epsilon$.*

Proof. Fix $f \in C[a, b]$, and choose $[a, b] \subseteq (-R, R)$. Extend f to a function $g \in C_0(\mathbb{R})$ on $\mathbb{R}$ which equals f on $[a, b]$ and is supported in $(-R, R)$. Let ϕ be the Gauss kernel, and choose λ so that

$$\|g - g * \phi_\lambda\|_\infty < \frac{\epsilon}{2}.$$

Note that $\phi_\lambda(x) = \lambda e^{-\pi\lambda^2 x^2}$ is analytic and has a Taylor expansion

$$\phi_\lambda(x) = \sum_{n=0}^{\infty} \lambda \frac{(-\pi\lambda^2 x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}.$$

Set $q(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}$. If N is large enough, then we will have

$$\sup_{|x| \leq 2R} |\phi_\lambda(x) - q(x)| < \frac{\epsilon}{2\|g\|_1}$$

since the Taylor series converges uniformly on compact sets. Then

$$|(g * \phi_\lambda)(x) - (g * q)(x)| \leq \int_{-R}^R |g(y)| |\phi_\lambda(x - y) - q(x - y)| dy \leq \frac{\epsilon}{2\|g\|_1} \int_{-R}^R |g(y)| dy < \frac{\epsilon}{2}$$

for $|x| \leq R$. So $|f(x) - (g * q)(x)| \leq \epsilon$ for $x \in [a, b]$. Finally, observe that

$$(g * q)(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} \int g(y) (x - y)^{2n} dy,$$

which is a polynomial by the binomial theorem. So we can take $p = g * q$. □

9.2 Schwartz Space

Definition 9.1. Define the *Schwartz space* $\mathcal{S}(\mathbb{R})$ to be

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for all } m, n \geq 0\}.$$

Remark. Note that $|x^m f^{(n)}(x)| \leq C_{m,n}$ for some constant $C_{m,n}$, so

$$|f^{(n)}(x)| \leq \frac{C_{m,n}}{|x|^m}.$$

In particular, for every n and polynomial p , there exists a constant $C_{n,p}$ such that

$$|f^{(n)}(x)| \leq \frac{C_{n,p}}{|p(x)|}.$$

Note, however, that this does not imply f has exponential decay.

Remark. Note that $\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty$ defines a *seminorm* for each $m, n \geq 0$, but not a norm.¹

¹Recall that a *seminorm* ρ is a function satisfying $0 \leq \rho(f) < \infty$, $\rho(cf) = |c|\rho(f)$, and $\rho(f + g) \leq \rho(f) + \rho(g)$. If we additionally have $\rho(f) = 0$ if and only if $f = 0$, then ρ is a *norm*.

Remark. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned} \|x^m f^{(n)}\|_1 &= \int_{|x| \leq 1} |x^m f^{(n)}(x)| dx + \int_{|x| \geq 1} |x^m f^{(n)}(x)| dx \\ &\leq 2\|x^m f^{(n)}\|_\infty + \int_{|x| \geq 1} \frac{|x^{m+2} f^{(n)}(x)|}{|x|^2} dx \leq 2\|x^m f^{(n)}\|_\infty + C\|x^{m+2} f^{(n)}\|_\infty. \end{aligned}$$

In particular, if the L^∞ -norms are controlled, then so are the L^1 -norms.

Exercise 9.3. Recall the smoothness and decay theorems: The Fourier transform interchanges smoothness and decay. Write $Df = f'$, and show that

$$(D^n((-2\pi i x)^m f(x)))^\wedge(\xi) = (2\pi i \xi)^n D^m \hat{f}(\xi).$$

Note that we have

$$D^n((-2\pi i x)^m f(x)) = \sum_{j=0}^n \binom{n}{j} D^j (-2\pi i x)^m f^{(n-j)}(x),$$

so in particular, the Schwartz condition on f implies the Schwartz condition on $\hat{f}$.

Theorem 9.2. If $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$. Moreover, we have $f = (\hat{f})^\vee$ by the inversion formula, so the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ by $f \mapsto \hat{f}$ is a bijection.

Remark. Note that $C_c^\infty(\mathbb{R}) \subsetneq \mathcal{S}(\mathbb{R})$, and we will see later that $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R}) \setminus C_c^\infty(\mathbb{R})$.

Lecture 10

Sept. 30 —

Lecture 11

Oct. 2 —

Lecture 12

Oct. 9 —

Lecture 13

Oct. 14 —

Lecture 14

Oct. 16 —

Lecture 15

Oct. 23 — The Paley-Wiener Theorem

15.1 Local Uncertainty Principle

Remark. Recall the classical uncertainty principle: For $f \in L^2(\mathbb{R})$, $\|f\|_2^2 \leq 4\pi \|xf(x)\|_2^2 \|\widehat{f}(\xi)\|_2^2$. There is also a local version of this statement:

Theorem 15.1 (Local uncertainty principle). *If $f \in L^2(\mathbb{R})$ and $\epsilon > 0$, then for any $\xi_0 \in \mathbb{R}$, we have*

$$\int_{\xi_0 - \epsilon}^{\xi_0 + \epsilon} |\widehat{f}(\xi)|^2 d\xi \leq 8\pi\epsilon \|f\|_2 \|xf(x)\|_2.$$

15.2 The Paley-Wiener Theorem

Remark. Recall the smoothness and decay theorems: If $x, xf(x) \in L^1(\mathbb{R})$, then $(\widehat{f})'$ exists. “Extreme decay” corresponds to compact support, where $\text{supp}(f) \subseteq [-T, T]$ if $f = 0$ a.e. outside $[-T, T]$. When this is the case, the *Paley-Wiener theorem* says that $\widehat{f}$ has extreme smoothness: $\widehat{f}$ extends to an analytic function on $\mathbb{C}$.

Remark. There is also motivation from Fourier series. The Fourier transform on $\mathbb{R}$ is

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx.$$

The functions $e_\xi(x) = e^{2\pi i \xi x} : \mathbb{R} \rightarrow S^1 \subseteq \mathbb{C}$ are continuous group homomorphism, and dx is a Haar measure on $\mathbb{R}$ (invariant under the group action). For $\mathbb{Z}$, Haar measure is the counting measure, and for $c = (c_n)_{n \in \mathbb{Z}} \in \ell^1$, we can define its Fourier transform to be

$$\widehat{c}(\xi) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n \xi}, \quad \xi \in \mathbb{R}.$$

In fact, we can take $\xi \in [0, 1)$. Also note that $\widehat{c}$ is 1-periodic.

Suppose now that c is compactly supported, say only $c_1, \dots, c_N$ are nonzero. Then

$$\widehat{c}(\xi) = \sum_{n=1}^N c_n e^{-2\pi i n \xi} = \sum_{n=1}^N c_n (e^{-2\pi i \xi})^n$$

If we set $z = e^{-2\pi i\xi} \in S^1$, then the above becomes

$$\widehat{c}(z) = \sum_{n=1}^N c_n z^n, \quad |z| = 1,$$

which is a polynomial. In particular, $\widehat{c}$ extends to an analytic function for all $z \in \mathbb{C}$.

Remark. Suppose $f \in L^2(\mathbb{R})$ and $\text{supp}(f) \subseteq [-T, T]$ (we say that f is “time-limited” to $[-T, T]$). Then

$$\int_{-\infty}^{\infty} |f(x)| dx = \int_{-T}^T |f(x)| dx \leq \left(\int_{-T}^T |f(x)|^2 dx \right)^{1/2} \left(\int_{-T}^T 1^2 dx \right)^{1/2} = (2T)^{1/2} \|f\|_2 < \infty,$$

so $f \in L^1(\mathbb{R})$. Thus we can take its Fourier transform as usual:

$$\widehat{f}(\xi) = \int_{-T}^T f(x) e^{-2\pi i\xi x} dx, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that $e^{-2\pi i\xi x} = e^{-2\pi i\alpha x} e^{2\pi\beta x}$, where $e^{2\pi\beta x} > 0$. Then

$$\int_{-T}^T |f(x) e^{-2\pi i\xi x}| dx = \int_{-T}^T |f(x)| e^{2\pi\beta x} dx \leq e^{2\pi|\beta|T} \int_{-T}^T |f(x)| dx \leq (2T)^{1/2} e^{2\pi|\beta|T} \|f\|_2.$$

So $\widehat{f}$ can be extended to all of $\mathbb{C}$. We will see that this extension is analytic.

We now have a function $\widehat{f} : \mathbb{C} \rightarrow \mathbb{C}$ satisfying

$$|\widehat{f}(\xi)| \leq (2T)^{1/2} e^{2\pi|\beta|T} \|f\|_2, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that $f \in L^1(\mathbb{R})$ implies $\widehat{f} \in C_0(\mathbb{R})$, so $\widehat{f}$ is bounded on $\mathbb{R}$. However, the extension to $\mathbb{C}$ cannot be bounded (for non-constant f) by Liouville’s theorem.

Definition 15.1. A function $F : \mathbb{C} \rightarrow \mathbb{C}$ has *exponential type* T if there exists $A > 0$ such that

$$|F(\xi)| \leq A e^{T|\xi|}, \quad \xi \in \mathbb{C}.$$

Remark. So from the above discussion, $\widehat{f}$ has exponential type $2\pi T$. We now show that $\widehat{f}$ is analytic on $\mathbb{C}$. Recall that F is analytic if and only if F' exists on $\mathbb{C}$. We can compute

$$(\widehat{f})'(\xi) = \lim_{\substack{\eta \rightarrow 0 \\ \xi \in \mathbb{C}}} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \rightarrow 0} \int_{-T}^T f(x) \frac{e^{-2\pi i(\xi + \eta)x} - e^{-2\pi i\xi x}}{\eta} dx.$$

We have seen that f is integrable, and the exponential term is bounded on $[-T, T]$ by the mean value theorem. So by the dominated convergence theorem, we can exchange the limit and integral to get

$$\begin{aligned} (\widehat{f})'(\xi) &= \int_{-T}^T f(x) \lim_{\eta \rightarrow 0} \frac{e^{-2\pi i(\xi + \eta)x} - e^{-2\pi i\xi x}}{\eta} dx \\ &= \int_{-T}^T (-2\pi i x) f(x) e^{-2\pi i\xi x} dx = ((-2\pi i x) f(x))^\wedge(\xi). \end{aligned}$$

Note that $x f(x) \in L^1(\mathbb{R})$ and also has support in $[-T, T]$, so the above exists for any $\xi \in \mathbb{C}$.

Theorem 15.2 (Paley-Wiener theorem). *We have the following:*

1. *If $f \in L^2(\mathbb{R})$ and $\text{supp}(f) \subseteq [-T, T]$, then $\widehat{f}$ extends to an analytic function on $\mathbb{C}$ with exponential type $2\pi T$.*
2. *If $F : \mathbb{C} \rightarrow \mathbb{C}$ is analytic with exponential type $2\pi T$, then there exists $f \in L^2(\mathbb{R})$ with $\text{supp}(f) \subseteq [-T, T]$ and $\widehat{f}(\xi) = F(\xi)$ for $\xi \in \mathbb{R}$, i.e. $\widehat{f} = F|_{\mathbb{R}}$.*

Proof. (1) This was the above discussion.

(2) The proof is more difficult and uses a lot of complex analysis. See Katznelson's book. $\square$

Remark. Recall that analytic functions have Taylor expansions. Suppose $F : \mathbb{C} \rightarrow \mathbb{C}$ and $F = 0$ on a line segment. Then if η lies on the line segment, we can write

$$F(\xi) = \sum_{k=0}^{\infty} F^{(k)}(\eta) \frac{(\xi - \eta)^k}{k!}.$$

Since $F^{(k)}(\eta) = 0$ for all k , we get $F \equiv 0$ on all of $\mathbb{C}$. In fact, if $F = 0$ on any set with an accumulation point, then $F \equiv 0$ on $\mathbb{C}$.

Corollary 15.2.1. *If $f \in L^2(\mathbb{R})$, $f \neq 0$, has compact support, then $\widehat{f}(\xi) = 0$ for only countably many ξ .*

Proof. If $\widehat{f}(\xi) = 0$ for uncountably many ξ , then there exists $\xi_n \rightarrow \xi$ such that $\widehat{f}(\xi_n) = 0$ for all n (split $\mathbb{C}$ into countably many compact sets and apply a countability argument). Hence $\widehat{f}, f = 0$. $\square$

Corollary 15.2.2. *If $f \in L^2(\mathbb{R})$, then f and $\widehat{f}$ are both compactly supported if and only if $f = 0$.*

Proof. Suppose f is compactly supported, then $\widehat{f}$ is analytic on $\mathbb{C}$. Suppose $\widehat{f}(\xi) = 0$ a.e. on $\mathbb{R}$ outside $[-\Omega, \Omega]$. Since $\widehat{f}$ is analytic and $\mathbb{R} \setminus [-\Omega, \Omega]$ has an accumulation point, we get $\widehat{f}, f = 0$. $\square$

Remark. In particular, the above corollary implies that if $f \in C_c^\infty(\mathbb{R})$ is nonzero, then $\widehat{f}$ is not compactly supported. But $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ and $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$, so we do know $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R})$.

15.3 Hilbert-Schmidt Operators

Remark. Let H be a separable, infinite-dimensional Hilbert space. Recall the following definitions and results from functional analysis:

Definition 15.2. A linear operator $A : H \rightarrow H$ is *compact* if $A(D)$ is contained in a compact set, where $D = \{f \in H : \|f\| \leq 1\}$ is the closed unit disk in H .

Theorem 15.3 (Spectral theorem for compact, self-adjoint operators). *If $A : H \rightarrow H$ is a compact, self-adjoint operator, then there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of H such that*

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n, \quad Af = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n, \quad Ae_n = \lambda_n e_n.$$

Moreover, $\lambda_n \rightarrow 0$ if there are infinitely many $\lambda_n \neq 0$.

Remark. Note that $\|Ae_n\| = \|\lambda_n e_n\| = |\lambda_n| \rightarrow 0$, so the identity operator I is not compact.

Example 15.2.1. An example of a compact box in $\mathbb{R}^{\mathbb{N}}$ looks like

$$[0, 1] \times [0, 1/2] \times [0, 1/4] \times \cdots .$$

Definition 15.3. A linear operator $A : H \rightarrow H$ is *Hilbert-Schmidt* if there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$.

Remark. For an operator $A : H \rightarrow H$, if we write

$$Af(x) = \int_{-\infty}^{\infty} k(x, y) f(y) dy,$$

then A is Hilbert-Schmidt if and only if $k \in L^2(\mathbb{R}^2)$.

Lecture 16

Oct. 28 —

Lecture 17

Oct. 30 —

Lecture 18

Nov. 4 — Energy Concentration, Part 2

18.1 Energy Concentration, Continued

Remark. Recall the following problem. Fix $T, \Omega > 0$. We want to determine

$$E_{T,\Omega} = \sup \left\{ \int_{-T}^T |f|^2 : \|f\|_2 = 1, \text{supp}(\widehat{f}) \subseteq [-\Omega, \Omega] \right\}.$$

We previously defined $A_T f = f \chi_{[-T,T]}$ and $B_\Omega f = (\widehat{f \chi_{[-\Omega,\Omega]}})^\vee = f * d_{2\pi\Omega}$, then

$$E_{T,\Omega} = \|A_T B_\Omega\|^2 = \|B_\Omega A_T B_\Omega\|^2 = \lambda_1,$$

where λ_1 is the largest eigenvalue of $B_\Omega A_T B_\Omega$ (recall that $B_\Omega A_T B_\Omega$ is compact and self-adjoint, so it has eigenvalues $\lambda_n \rightarrow 0$). The spectral theorem also gives orthonormal eigenvectors $\{\varphi_n\}_{n \in \mathbb{N}}$:

$$B_\Omega A_T B_\Omega f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi_n.$$

Note that $B_\Omega(A_T B_\Omega \varphi_n) = \lambda_n \varphi_n$ for $\lambda_n \neq 0$, so $\varphi_n \in \mathcal{FL}_{[-\Omega,\Omega]}^2(\mathbb{R})$ and $\text{supp}(\widehat{\varphi_n}) \subseteq [-\Omega, \Omega]$.

What is $\|A_T \varphi_n\|_2^2$? We can compute

$$\begin{aligned} \|A_T \varphi_n\|_2^2 &= \langle A_T \varphi_n, A_T \varphi_n \rangle = \langle A_T B_\Omega \varphi_n, A_T B_\Omega \varphi_n \rangle \\ &= \langle B_\Omega \underbrace{A_T A_T}_{A_T} B_\Omega \varphi_n, \varphi_n \rangle = \lambda_n \langle \varphi_n, \varphi_n \rangle = \lambda_n \|\varphi_n\|_2^2 = \lambda_n. \end{aligned}$$

Thus we see that $E_{T,\Omega} = \lambda_1 = \|A_T \varphi_1\|_2^2 < \|\varphi_1\|_2^2 = 1$, so φ_1 has the greatest energy in $[-T, T]$.

Note that $\{\varphi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for its closed span, which a priori lies in $\mathcal{FL}_{[-\Omega,\Omega]}^2(\mathbb{R})$.

Proposition 18.1. $\overline{\text{span}}\{\varphi_n\} = \mathcal{FL}_{[-\Omega,\Omega]}^2(\mathbb{R})$.

Proof. We show $\overline{\text{span}}\{\varphi_n\}^\perp = \{0\}$. Suppose $f \in \mathcal{FL}_{[-\Omega,\Omega]}^2(\mathbb{R})$ with $f \perp \varphi_n$ for all n . Then

$$f \in \overline{\text{span}}\{\varphi_n\}^\perp = \overline{\text{range}}(B_\Omega A_T B_\Omega)^\perp = \ker(B_\Omega A_T B_\Omega).$$

Thus we have

$$\|A_T f\|_2^2 = \langle A_T B_\Omega f, A_T B_\Omega f \rangle = \langle B_\Omega A_T B_\Omega f, f \rangle = 0,$$

so $f = 0$ a.e. on $[-T, T]$. By the Paley-Wiener theorem, this implies $f = 0$. □

Remark. Proposition 18.1 implies that there are infinitely many nonzero eigenvalues for $B_\Omega A_T B_\Omega$.

Moreover, since B_Ω is the orthogonal projection onto $\mathcal{FL}_{[-\Omega, \Omega]}^2(\mathbb{R})$ and $\{\varphi_n\}$ is an orthonormal basis for $\mathcal{FL}_{[-\Omega, \Omega]}^2(\mathbb{R})$, we can write

$$B_\Omega f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n.$$

Note that $L_{[-T, T]}^2(\mathbb{R})$ is not orthogonal to $\mathcal{FL}_{[-\Omega, \Omega]}^2(\mathbb{R})$: For example,

$$\langle \chi_{[-T, T]}, d_{2\pi\Omega} \rangle = \int_{-T}^T \frac{\sin 2\pi\Omega x}{\pi x} dx \neq 0.$$

However, set $\psi_n = \lambda_n^{-1/2} A_T \varphi_n$. Then

$$\langle \lambda_m^{-1/2} A_T \varphi_m, \lambda_n^{-1/2} A_T \varphi_n \rangle = \lambda_m^{-1/2} \lambda_n^{-1/2} \langle B_\Omega A_T B_\Omega \varphi_m, \varphi_n \rangle = \lambda_m^{1/2} \lambda_n^{-1/2} \langle \varphi_m, \varphi_n \rangle = \delta_{mn}.$$

So $\{\psi_n\}_{n \in \mathbb{N}}$ is an orthonormal sequence in $L_{[-T, T]}^2(\mathbb{R})$.

Exercise 18.1. Show that $\{\psi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for $L_{[-T, T]}^2(\mathbb{R})$.

Remark. Note that

$$\langle \varphi_m, \varphi_n \rangle = \int_{-\infty}^{\infty} \varphi_m(x) \overline{\varphi_n(x)} dx = 0, \quad m \neq n.$$

Furthermore,

$$\langle \psi_m, \psi_n \rangle = \lambda_m^{-1/2} \lambda_n^{-1/2} \int_{-T}^T \varphi_m(x) \overline{\varphi_n(x)} dx = 0, \quad m \neq n$$

even when the φ_n are *not* supported in $[-T, T]$.

By symmetry (swapping the roles of A_T and B_Ω), one can also compute that

$$A_T B_\Omega A_T f = \sum_{n=1}^{\infty} \lambda_n \langle f, \psi_n \rangle \psi_n.$$

In particular, $A_T B_\Omega A_T$ and $B_\Omega A_T B_\Omega$ have the same eigenvalues.

Proposition 18.2. $B_\Omega A_T$ commutes with

$$Kf = (T^2 - x^2)f''(x) - 2xf'(x) - 4\pi^2\Omega^2 x^2 f(x).$$

Remark. Since φ_n is already band-limited, we have

$$B_\Omega A_T \varphi_n = B_\Omega A_T B_\Omega \varphi_n = \lambda_n \varphi_n.$$

Moreover, Paley-Wiener implies φ_n is infinitely differentiable, so $\varphi_n \in \text{domain}(K)$. Then

$$B_\Omega A_T K \varphi_n = K B_\Omega A_T \varphi_n = K(\lambda_n \varphi_n) = \lambda_n K \varphi_n.$$

Note that $K \varphi_n \neq 0$, so $K \varphi_n$ is an eigenvector for $B_\Omega A_T$. The multiplicities of the λ_n is 1, so $K \varphi_n = \mu \varphi_n$. The eigenfunctions φ_n of K are known functions called the *prolate spheroidal wave functions*.

18.2 Approximating Band-Limited Functions

Remark. We know that $1 > \lambda_1 > \lambda_2 > \dots \rightarrow 0$. Also,

$$\|A_T B_\Omega\|_{\text{HS}}^2 = \|g_{T,\Omega}\|_2^2 = \int_{-T}^T \int_{-\infty}^{\infty} d_{2\pi\Omega}(x-y)^2 dx dy = 4T\Omega,$$

where $g_{T,\Omega}$ is the corresponding kernel. The singular values of $A_T B_\Omega$ are

$$s_n = \lambda_n((A_T B_\Omega)^*(A_T B_\Omega))^{1/2} = \lambda_n(B_\Omega A_T B_\Omega)^{1/2} = \lambda_n^{1/2}.$$

Thus $4T\Omega = \|A_T B_\Omega\|_{\text{HS}}^2 = \sum_{n=1}^{\infty} s_n^2 = \sum_{n=1}^{\infty} \lambda_n$. On the other hand, $\|B_\Omega A_T B_\Omega\|_{\text{HS}}^2 = \sum_{n=1}^{\infty} \lambda_n^2$, and

$$\|B_\Omega A_T B_\Omega\|_{\text{HS}}^2 = \|k_{T,\Omega}\|_2^2 = \int_{-T}^T \int_{-T}^T d_{2\pi\Omega}(x-y)^2 dx dy < 4T\Omega.$$

Due to the decay of $d_{2\pi\Omega}(x-y)^2$, for T large, we should expect that $\|B_\Omega A_T B_\Omega\|_{\text{HS}}^2 \rightarrow 4T\Omega$. Hence

$$\sum_{n=1}^{\infty} \lambda_n^2 \approx \sum_{n=1}^{\infty} \lambda_n.$$

Thus intuitively, λ_n should be either close to 1 or close to 0. In particular, only finitely many terms can be close to 1, so

$$f \approx \sum_{n=1}^N \lambda \langle f, \varphi_n \rangle \varphi_n \in \text{span}\{\varphi_1, \dots, \varphi_N\}.$$

Thus the space of band-limited functions are approximately finite-dimensional in some sense.

Theorem 18.1. $\#\{n : \lambda_n \geq 1 - \epsilon\} \leq 4T\Omega - C_\epsilon \log(T\Omega)$.

18.3 Space of Possible Time and Band Limits

Remark. Define the energies

$$\begin{aligned} \alpha &= E_T(f) = \|A_T f\|_2^2 = \int_{-T}^T |f|^2 \\ \beta &= E_\Omega(f) = \|B_\Omega f\|_2^2 = \int_{-\Omega}^{\Omega} |\hat{f}|^2. \end{aligned}$$

Which pairs (α, β) can be achieved? We have seen that $(\lambda_1, 1)$ and $(\lambda_2, 1)$ are possible, while $(1, 1)$ is not by the Paley-Wiener theorem. Similarly, $(1, 0)$ and $(0, 1)$ are not possible by Paley-Wiener. Since the roles of A_T and B_Ω are symmetric, we get that $(1, \lambda_1)$ and $(1, \lambda_2)$ are possible. Note that if $(\alpha, 1)$ is achieved, then so is $(\alpha', 1)$ for any $\alpha' < \alpha$, and a similar statement holds for $(1, \beta)$.

One can show the space of valid pairs is $(\alpha, \beta) \in [0, 1]^2$ such that

$$\cos^{-1} \alpha + \cos^{-1} \beta \geq \cos^{-1} \lambda^{1/2},$$

which is in the shape of an ellipse.

Lecture 19

Nov. 6 — Fourier Series

19.1 The Abstract Fourier Transform

Definition 19.1. A *locally compact abelian (LCA) group* is an abelian group with a Hausdorff topology such that every point has a compact neighborhood, and $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous.

Example 19.1.1. Some examples of LCA groups are $\mathbb{R}^n$, $\mathbb{Z}^n$, $\mathbb{T}^n$, $\mathbb{Z}_N^n$, where $\mathbb{T} = [0, 1)$ with addition modulo 1.

Theorem 19.1. Every LCA group has a Haar measure, i.e. a nonzero Radon measure μ which is translation-invariant.

Definition 19.2. A *character* on a LCA group is a continuous homomorphism

$$\xi : G \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

i.e. a continuous map with $\xi(x + y) = \xi(x)\xi(y)$.

Definition 19.3. The *dual group* of G is $\widehat{G} = \{\xi : \xi \text{ is a character of } G\}$.

Example 19.3.1. Consider $\mathbb{T} = [0, 1)$. Fix $n \in \mathbb{Z}$, then $e_n(x) = e^{2\pi i n x}$ is a character on $\mathbb{T}$. Note that we have $e_n(x) = e_1(x)^n$, and one can show these are all of the characters on $\mathbb{T}$. Thus

$$\widehat{\mathbb{T}} = \{e_n : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Example 19.3.2. The characters on $\mathbb{Z}$ are determined by their values at 1. So for any $\xi \in [0, 1)$, we get a character $e_\xi(n) = e^{2\pi i \xi n}$. In particular, $\widehat{\mathbb{Z}} = \mathbb{T}$.

Example 19.3.3. We have $\widehat{\mathbb{Z}_N} \cong \mathbb{Z}_N$ and $\widehat{\mathbb{R}} \cong \mathbb{R}$. The characters on $\mathbb{R}$ are $e_\xi(x) = e^{2\pi i \xi x}$.

Remark. The dual group $\widehat{G}$ is a LCA group under the multiplication of characters, where the topology on $\widehat{G}$ is the topology of uniform convergence on compact sets (also called the compact-open topology).

Theorem 19.2. If G is discrete, then $\widehat{G}$ is compact. If G is compact (so G has finite measure), then $\widehat{G}$ is discrete, and the characters on G are orthonormal in $L^2(G)$ (in fact, they form an orthonormal basis).

Example 19.3.4. Consider the examples $\widehat{\mathbb{Z}} = \mathbb{T}$, $\widehat{\mathbb{T}} = \mathbb{Z}$, $\widehat{\mathbb{R}} = \mathbb{R}$, $\widehat{\mathbb{Z}_N} = \mathbb{Z}_N$.

Remark. In some sense, $\mathbb{R}$ being non-compact and non-discrete makes things harder. On the other hand, having dilations in $\mathbb{R}$ can make harmonic analysis easier.

Theorem 19.3 (Pontryagin duality). $\widehat{\widehat{G}} \cong G$.

Definition 19.4. The *Fourier transform* of $f \in L^1(G)$ (complex-valued) is $\widehat{f}: \widehat{G} \rightarrow \mathbb{C}$ defined by

$$\widehat{f}(\xi) = \int_G f(x) \overline{\xi(x)} d\mu(x), \quad \xi \in \widehat{G},$$

where $d\mu$ is the Haar measure on G .

Example 19.4.1. On the real line, the Fourier transform is

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx, \quad \xi \in \mathbb{R}.$$

For $\mathbb{T}$, the Fourier coefficients are given by

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx, \quad n \in \mathbb{Z}.$$

For $\mathbb{Z}$, given $c = (c_n)_{n \in \mathbb{Z}}$,

$$\widehat{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n x}, \quad x \in \mathbb{T}$$

For $\mathbb{Z}_N$, the discrete Fourier transform is

$$\widehat{c}(k) = \sum_{n=0}^{N-1} c_n e^{-2\pi i n k / N}, \quad k \in \mathbb{Z}_N.$$

19.2 Fourier Series

Definition 19.5. The *Fourier transform* of $f \in L^1(\mathbb{T})$ is the sequence $\widehat{f} = (\widehat{f}(n))_{n \in \mathbb{Z}}$, where

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$$

are the *Fourier coefficients* of f . The *inverse Fourier transform* of $f \in L^1(\mathbb{T})$ is $\check{f} = (\check{f}(n))_{n \in \mathbb{Z}}$, where

$$\check{f}(n) = \int_0^1 f(x) e^{2\pi i n x} dx.$$

Formally, $\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$ is the *Fourier series* of f .

Remark. If $f \in L^2(\mathbb{T})$ and we accept that $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n,$$

where the convergence is in $L^2(\mathbb{T})$ (we do not get pointwise convergence in general).

Remark. If $f \in L^1(\mathbb{T})$, then we have

$$|\widehat{f}(n)| \leq \int_0^1 |f(x) e^{-2\pi i n x}| dx = \|f\|_{L^1},$$

so we see that $\widehat{f} \in \ell^\infty(\mathbb{Z})$.

Remark. If we define $\check{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$, then we can view the Fourier series as

$$(\widehat{f})^\vee(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

Example 19.5.1. For $0 < \alpha \leq 1$, consider the function

$$g(x) = \sum_{m=0}^{\infty} 2^{-\alpha m} e^{2\pi i 2^m x}, \quad x \in \mathbb{T}$$

which converges absolutely in $C(\mathbb{T})$. One can show that g is nowhere differentiable.

Weierstrass first showed that $g(x) = \sum_{m=0}^{\infty} a^{-m} \cos(b^m x)$ is nowhere differentiable for a, b large enough.

19.3 Partial Sums of Fourier Series

Definition 19.6. For $f \in L^1(\mathbb{T})$, define the *partial sums*

$$S_N f(x) = \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x}.$$

Remark. Consider a discrete characteristic function

$$\chi_N(n) = \begin{cases} 1 & |n| \leq N, \\ 0 & |n| > N. \end{cases}$$

Note that $\chi_N \in \ell^1(\mathbb{Z})$. Then we can compute that

$$\widehat{\chi}_N(x) = \check{\chi}_N(x) = \sum_{n=-\infty}^{\infty} \chi_N(n) e^{2\pi i n x} = \sum_{n=-N}^N e^{2\pi i n x} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} = d_N(x)$$

by the geometric series formula. Note that $d_N \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$.¹ We also have $\|d_N\|_1 \rightarrow \infty$.

Remark. Using the *Dirichlet kernel* d_N , we can write the partial sums as

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x} = \sum_{n=-N}^N \left(\int_0^1 f(t) e^{-2\pi i n t} dt \right) e^{2\pi i n x} \\ &= \int_0^1 f(t) \sum_{n=-N}^N e^{2\pi i n(x-t)} dt = \int_0^1 f(t) d_N(x-t) dt = (f * d_N)(x). \end{aligned}$$

Intuitively, $S_N f = (\widehat{f} \chi_N)^\vee = (\widehat{f})^\vee * \check{\chi}_N = f * d_N$, but the above is a direct proof.

¹Note that $\mathbb{T}$ is compact, so a continuous function on $\mathbb{T}$ is automatically integrable.

19.4 Approximate Identities on $\mathbb{T}$

Definition 19.7. An *approximate identity* on $\mathbb{T}$ is a sequence $\{k_n\}_{n \in \mathbb{N}}$ such that

1. $k_n \in L^1(\mathbb{T})$ and $\int_0^1 k_n(x) dx = 1$;
2. $\sup_{n \in \mathbb{N}} \|k_n\|_1 < \infty$;
3. for any $\delta > 0$, $\int_{|x| \geq \delta} |k_n(x)| dx \rightarrow 0$.

Remark. The Dirichlet kernel d_N does not form an approximate identity.

Definition 19.8. For $f \in L^1(\mathbb{T})$, define the *Cesàro sums*

$$\sigma_N f(x) = \frac{S_0 f(x) + S_1 f(x) + \cdots + S_N f(x)}{N+1}.$$

Note that this is the average of the partial sums.

Remark. We can rewrite the Cesàro sums as

$$\sigma_N f(x) = \sum_{n=-N}^N \underbrace{\left(1 - \frac{|n|}{N+1}\right)}_{W_N(n)} \widehat{f}(n) e^{2\pi i n x} = (f * \widehat{W}_N)(x).$$

We define the *Fejér kernel* to be

$$w_N(x) = \widehat{W}_N(x) = \frac{1}{N+1} \left(\frac{\sin((N+1)\pi x)}{\sin(\pi x)} \right)^2.$$

Lecture 20

Nov. 11 — Fourier Series, Part 2

20.1 Approximate Identities on $\mathbb{T}$, Continued

Remark. Using a similar proof as in the case for $\mathbb{R}$, for $f \in L^1(\mathbb{T})$, we have

$$\widehat{f} \in c_0(\mathbb{Z}) = \{c = (c_n) : c_n \rightarrow 0 \text{ as } |n| \rightarrow \infty\}.$$

However, in general $\widehat{f} \notin \ell^1(\mathbb{Z})$.

Exercise 20.1. Let d_N be the Dirichlet kernel on $\mathbb{T}$. Show that

$$\frac{4}{\pi^2} \sum_{k=1}^N \frac{1}{k} \leq \|d_N\|_1 \leq 3 + \frac{4}{\pi^2} \sum_{k=1}^N \frac{1}{k}.$$

In particular, $\|d_N\|_1 \rightarrow \infty$ as $N \rightarrow \infty$, so d_N does not form an approximate identity.

Remark. Define the function

$$W_N(x) = \begin{cases} 1 - |n|/(N+1) & |n| \leq N, \\ 0 & |n| > N. \end{cases}$$

Note that $W_N = (\chi_N * \chi_N)/(2N+1)$ and $\widehat{W}_{2N} = (\widehat{\chi}_N)^2 = d_N^2/(2N+1)$. The Fejér kernel is

$$w_N = \widehat{W}_N = \frac{1}{N} \left(\frac{\sin((N+1)\pi x)}{\sin(\pi x)} \right)^2,$$

and one can check that $\int_0^1 w_N = 1$ and $\|w_N\|_1 = 1$.

Theorem 20.1. $(f * \widetilde{W}_N)(x) = (f * w_N)(x)$.

Exercise 20.2. $\{w_N\}_{N \in \mathbb{N}}$ forms an approximate identity on $\mathbb{T}$.

Remark. Unlike the real line, we cannot just take any L^1 function and dilate it to form an approximate identity on $\mathbb{T}$, as the dilations need not be 1-periodic.

Exercise 20.3. If $\{k_N\}$ is an approximate identity on $\mathbb{T}$, then:

1. For $1 \leq p < \infty$, $f \in L^p(\mathbb{T})$ implies $f * k_N \rightarrow f$ in L^p -norm.
2. $f \in C(\mathbb{T})$ implies $f * k_N \rightarrow f$ uniformly.

20.2 The Inversion Formula on $\mathbb{T}$

Theorem 20.2. *If $f \in L^1(\mathbb{T})$ and $\widehat{f} \in \ell^1(\mathbb{Z})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x},$$

where the above series converges uniformly in $C(\mathbb{T})$.

Proof. Note that $(\widehat{f})^\vee(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$ converges absolutely in the norm of $C(\mathbb{T})$ since

$$\sum_{n=-\infty}^{\infty} \|\widehat{f}(n) e^{2\pi i n x}\|_\infty = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty.$$

In particular, this shows that $(\widehat{f})^\vee \in C(\mathbb{T})$. Also $f * w_N \rightarrow f$ in L^1 -form, where

$$(f * w_N)(x) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) e^{2\pi i n x}.$$

Note that $(1 - |n|/(N+1))\widehat{f}(n) \rightarrow \widehat{f}(n)$ pointwise as $N \rightarrow \infty$, and

$$\left| \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) \right| \leq |\widehat{f}(n)| \in \ell^1(\mathbb{Z}),$$

so by the dominated convergence theorem (for counting measure)

$$(f * w_N)(x) \xrightarrow{N \rightarrow \infty} \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x} = (\widehat{f})^\vee(x)$$

pointwise. Since $f * w_N \rightarrow f$ in L^1 -norm, there exists a subsequence $f * w_{N_k} \rightarrow f$ pointwise a.e., hence we see that $f = (\widehat{f})^\vee$ a.e. Since f and $(\widehat{f})^\vee$ are continuous, we get that $f = (\widehat{f})^\vee$ everywhere. $\square$

Corollary 20.2.1. *If $f \in L^1(\mathbb{T})$ and $\widehat{f}(n) = 0$ for every n , then $f = 0$ a.e. Moreover, if $f, g \in L^1(\mathbb{T})$ and $\widehat{f}(n) = \widehat{g}(n)$ for all n , then $f = g$ a.e.*

Definition 20.1. The *Fourier algebra* (or *Wiener algebra*) on $\mathbb{T}$ is

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f} \in \ell^1(\mathbb{Z})\}.$$

Remark. The Fourier algebra $A(\mathbb{T})$ is closed under convolution and forms a Banach algebra. Moreover, it is a dense subspace of $C(\mathbb{T})$.

20.3 L^2 -Convergence of Fourier Series

Remark. Let $e_n(x) = e^{2\pi i n x}$. The functions e_n are orthonormal in $L^2(\mathbb{T})$: For $n \neq m$,

$$\langle e_m, e_n \rangle = \int_0^1 e^{2\pi i m x} \overline{e^{2\pi i n x}} dx = \int_0^1 e^{2\pi i (m-n)x} dx = \frac{e^{2\pi i (m-n)x}}{2\pi i (m-n)} \Big|_{x=0}^1 = 0.$$

Theorem 20.3. Let $\{f_n\}_{n \in \mathbb{N}}$ be an orthonormal sequence in a Hilbert space H . Then the following are equivalent:

1. $\{f_n\}_{n \in \mathbb{N}}$ is complete (i.e. $\overline{\text{span}}\{f_n\} = H$).
2. $f = \sum_{n=1}^{\infty} c_n f_n$ in H for a unique choice of scalars c_n . That is, $\{f_n\}$ is a Schauder basis for H .
3. $f = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n$, where the convergence is in H .
4. (Plancherel) $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2$.
5. (Parseval) $\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, f_n \rangle \langle f_n, g \rangle$.

Example 20.1.1. The above theorem does not hold when the sequence is not orthonormal. Recall the Weierstrass approximation theorem: $\{x^n\}_{n=0}^{\infty}$ is complete in $C(\mathbb{T})$. But not every $f \in C(\mathbb{T})$ can be written as $f(x) = \sum_{n=0}^{\infty} c_n x^n$ (such functions are infinitely differentiable in some disk).

Theorem 20.4. We have the following:

1. For $1 \leq p < \infty$, $\{e_n\}_{n \in \mathbb{N}}$ is complete in $L^p(\mathbb{T})$.
2. $\{e_n\}_{n \in \mathbb{N}}$ is complete in $C(\mathbb{T})$.

Proof. (1) If $f \in L^p(\mathbb{T})$, then we have

$$(f * w_N) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) e^{2\pi i n x} \xrightarrow{N \rightarrow \infty} f$$

in L^p -norm. But $f * w_n \in \text{span}\{e_n\}_{n \in \mathbb{Z}}$, so $\overline{\text{span}}\{e_n\}_{n \in \mathbb{Z}} = L^p(\mathbb{T})$. The same proof works for (2). $\square$

Corollary 20.4.1. $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$. Furthermore,

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x},$$

where the above convergence is in L^2 -norm. Moreover, one has

1. (Plancherel) $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$,
2. (Parseval) $\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \overline{\widehat{g}(n)}$,

so the Fourier transform operator $\mathcal{F} : L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{Z})$ given by $f \mapsto \widehat{f}$ is unitary.

Theorem 20.5. If $1 < p < \infty$, then $S_N f = (f * d_N)(x) = \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x}$ converges in L^p -norm, so

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

in L^p -norm with respect to the ordering $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$. However, the convergence is conditional: Only this ordering (and finite permutations of it) need converge when $p \neq 2$.

Lecture 21

Nov. 13 — Fourier Series, Part 3

21.1 Shannon Sampling Theorem

Definition 21.1. Define the *Paley-Wiener space* to be

$$\text{PW}(\mathbb{R}) = \mathcal{FL}_{[-1/2, 1/2]}^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp}(\widehat{f}) \subseteq [-1/2, 1/2]\}.$$

Remark. Set $e_n(x) = e^{-2\pi i n x} \chi_{[-1/2, 1/2]}(x) = M_n \chi_{[-1/2, 1/2]}(x)$, then

$$\check{e}_n(x) = T_n \check{\chi}_{[-1/2, 1/2]}(x) = \frac{\sin(\pi(x - n))}{\pi(x - n)}.$$

Since $\{e_n\}$ is an orthonormal basis for $L_{[-1/2, 1/2]}^2(\mathbb{R})$, the $\{\check{e}_n\}$ form an orthonormal basis for $\text{PW}(\mathbb{R})$. Note that if $f \in \text{PW}(\mathbb{R})$, then we can write

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f, \check{e}_n \rangle \check{e}_n(x).$$

By the unitarity of the Fourier transform, we can compute that

$$\langle f, \check{e}_n \rangle = \langle \widehat{f}, e_n \rangle = \int_{-1/2}^{1/2} \widehat{f}(\xi) e^{2\pi i n \xi} d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i n \xi} d\xi = (\widehat{f})^\vee(n) = f(n).$$

Note that the third equality follows from $\text{supp}(\widehat{f}) \subseteq [-1/2, 1/2]$. Thus

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin(\pi(x - n))}{\pi(x - n)}.$$

This is called the *Shannon sampling theorem* (or *classical sampling theorem*), i.e. that any function in $\text{PW}(\mathbb{R})$ is completely determined by its values on the integers.

Remark. Fix $b > 0$, and set $e_{nb}(x) = e^{2\pi i n b x}$. This is an orthogonal basis for $L^2[0, 1/b]$.

Example 21.1.1. If we set $b = 1/2$, then we can write

$$\{e_{n/2}\}_{n \in \mathbb{Z}} = \{e_n\}_{n \in \mathbb{Z}} \cup \{e_{(n+1)/2}\}_{n \in \mathbb{Z}}.$$

This is a union of two orthonormal bases for $L^2[0, 1]$. For $f \in L^2[0, 1]$, we can write

$$2f = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n + \sum_{n=-\infty}^{\infty} \langle f, e_{(n+1)/2} \rangle e_{(n+1)/2} = \sum_{n \in \mathbb{Z}} \langle f, e_{n/2} \rangle e_{n/2}.$$

Remark. If we take $0 < b < 1$ and consider $e^{2\pi i n b x} = e_{bn}(x)$ in $L^2[0, 1]$, then for $f \in L^2[0, 1] \subseteq L^2[0, 1/b]$,

$$f = \frac{1}{b} \sum_{n=-\infty}^{\infty} \langle f, e_{bn} \rangle e_{bn}$$

in $L^2[0, 1/b]$. Since $f = 0$ on $[1, 1/b]$, we have the same expansion in $L^2[0, 1]$. Note, however, that this expansion is not orthogonal and not unique. These sets $\{e_{bn}\}_{n \in \mathbb{Z}}$ are called *frames*.

21.2 Weyl's Equidistribution Theorem

Theorem 21.1. If $\alpha \in \mathbb{T}$ is irrational, then $\{k\alpha\}_{k \in \mathbb{N}}$ is equidistributed in $[0, 1)$, i.e.

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq k \leq N : k\alpha \bmod 1 \in (a, b)\}}{N} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \chi_{(a,b)}(k\alpha) = b - a$$

for every $0 \leq a < b \leq 1$.

Remark. Consider $f \in C(\mathbb{T})$. Then we can view $\frac{1}{N} \sum_{k=1}^N f(k\alpha)$ as some type of Riemann sum, so we might expect that

$$\frac{1}{N} \sum_{n=1}^N f(k\alpha) \xrightarrow{N \rightarrow \infty} \int_0^1 f(x) dx.$$

The *Birkhoff ergodic theorem* says that the above average in fact converges to the integral (given that $\{k\alpha\}$ is equidistributed). In fact, the result holds for any $f \in L^1(\mathbb{T})$.

Exercise 21.1. Prove the above statement for $e_n(x) = e^{2\pi i n x}$ (and thus for trigonometric polynomials $p(x) = \sum_{n=-N}^N c_n e^{2\pi i n x}$). Then apply density to prove this for $f \in C(\mathbb{T})$.

Proof of Theorem 21.1. Approximate $\chi_{(a,b)}$ by a trapezoidal continuous function f which equals $\chi_{(a,b)}$ on $(a + \epsilon, b - \epsilon)$ and decays linearly to 0 on $[a, a + \epsilon]$ and $[b - \epsilon, b]$. Then

$$b - a - \epsilon = \int_0^1 f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(k\alpha) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \chi_{(a,b)}(k\alpha).$$

Taking a similar approximation with g which equals $\chi_{(a,b)}$ on $[a, b]$ and decays linearly to 0 on $[a - \epsilon, a]$ and $[b, b + \epsilon]$, we also get an upper bound:

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \chi_{(a,b)}(k\alpha) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N g(k\alpha) \leq \int_0^1 g(x) dx = b - a + \epsilon.$$

This holds for arbitrary ϵ , hence the limit is equal to $b - a$. □

Corollary 21.1.1 (Kronecker's theorem). $\{k\alpha \bmod 1\}_{k \in \mathbb{N}}$ is dense in $[0, 1)$.

21.3 L^p -Convergence of Fourier Series

Remark. Recall that $\{e_n\}_{n \in \mathbb{Z}}$ is complete in $L^p(\mathbb{T})$ and $C(\mathbb{T})$, i.e. $\text{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense.

For $1 \leq p < \infty$, if $f \in L^p(\mathbb{T})$, do the partial sums

$$S_N f(x) = \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x} = (f * d_N)(x)$$

converge to f in L^p -norm?

The motivation is as follows. Suppose that $S_N f \rightarrow f$ in L^p for every $f \in L^p(\mathbb{T})$. Then we have a linear operator $S_N : L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$ on a Banach space. Moreover, the convergence is also pointwise. Thus for any individual $f \in L^p(\mathbb{T})$,

$$\sup_{N \in \mathbb{N}} \|S_N f\|_p < \infty.$$

Then by the Banach-Steinhaus theorem (also called the uniform boundedness principle), we get

$$\sup_{N \in \mathbb{N}} \|S_N\| = \sup_{N \in \mathbb{N}} \sup_{\|f\|_p=1} \|S_N f\|_p < \infty$$

In fact, this is equivalent to the L^p -convergence of Fourier series, and idea behind the proof is to instead argue about the uniform boundedness of S_N .

Note that $\|S_N f\|_p = \|f * d_N\|_p \leq \|f\|_p \|d_N\|_1$, so $\|S_N\| \leq \|d_N\|_1 \rightarrow \infty$.

For $p = 1$, we can argue as follows. Fix an approximate identity $\{k_\lambda\}$ with $k_\lambda \geq 0$, so that

$$\|k_\lambda\|_1 = \int_0^1 |k_\lambda| = \int_0^1 k_\lambda = 1.$$

Then we can see that

$$\|S_N\| \geq \|S_N k_\lambda\|_1 = \|k_\lambda * d_N\|_1 \xrightarrow{\lambda \rightarrow \infty} \|d_N\|_1$$

since $k_\lambda * d_N \rightarrow d_N$ in L^1 . Thus $\|S_N\| \geq \|d_N\|_1 \rightarrow \infty$, so the partial sums cannot converge in L^1 . In particular, $\{e_n\}_{n \in \mathbb{N}}$ is *not* a Schauder basis for $L^1(\mathbb{T})$. A similar argument works to show that $\{e_n\}_{n \in \mathbb{N}}$ is not a Schauder basis for $C(\mathbb{T})$. However, one has the following:

Theorem 21.2. $\{e_n\}_{n \in \mathbb{Z}}$ is a Schauder basis for $L^p(\mathbb{T})$ for $1 < p < \infty$, with the ordering

$$\mathbb{Z} = \{0, -1, 1, -2, 2, \dots\}.$$

Example 21.1.2 (Haar system). Another example of a Schauder basis is the following. Let

$$\varphi = \chi_{[0,1)} \quad \text{and} \quad \psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}.$$

Note that $\langle \varphi, \psi \rangle = 0$. Then we can define

$$\psi_1 = 2^{1/2} \psi(2x) \quad \text{and} \quad \psi_2 = 2^{1/2} \psi(2x - 1),$$

which are still orthogonal with φ, ψ and with each other. Continuing this process, we get an orthonormal basis for $L^2(\mathbb{T})$. Moreover, this is a Schauder basis for $L^p(\mathbb{T})$ for $1 < p < \infty$ and it is unconditional, i.e. the convergence does not depend on the ordering of the functions.

Remark. There also exists a continuous analogue of the Haar system, the functions are called D_4 and W_4 . The functions φ, ψ from the Haar system are known as D_2, W_2 . These are called *wavelets*.

Lecture 22

Nov. 18 — Fourier Series, Part 4

22.1 The Poisson Summation Formula

Remark. Suppose $f \in L^1(\mathbb{R})$. Set $\varphi(x) = \sum_{n=-\infty}^{\infty} f(x+n)$. Then by Fubini-Tonelli,

$$\|\varphi\|_1 = \int_0^1 \left| \sum_{n=-\infty}^{\infty} f(x+n) \right| dx \leq \sum_{n=-\infty}^{\infty} \int_0^1 |f(x+n)| dx = \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f| = \int_{-\infty}^{\infty} |f| = \|f\|_1 < \infty,$$

so $\varphi \in L^1(\mathbb{T})$. Moreover, the integrals are the same if we do not take absolute values.

Exercise 22.1. Let f, φ as above. Prove that

$$\widehat{\varphi}(n) = \int_0^1 \sum_{n=-\infty}^{\infty} f(x) e^{-2\pi i n x} dx = \widehat{f}(n).$$

Theorem 22.1 (Poisson summation). *Let $f : \mathbb{R} \rightarrow \mathbb{C}$. If there exist $C, \epsilon > 0$ such that*

$$|f(x)| \leq \frac{C}{(1+|x|)^{1+\epsilon}} \quad \text{and} \quad |\widehat{f}(x)| \leq \frac{C}{(1+|x|)^{1+\epsilon}},$$

then for every $x \in \mathbb{R}$,

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

In particular, setting $x = 0$ one obtains

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

Proof. Both $f, \widehat{f} \in L^1(\mathbb{R})$, hence $f, \widehat{f} \in C_0(\mathbb{R})$ by Fourier inversion. Also $\varphi \in L^1(\mathbb{T})$ and

$$\sum_{n=-\infty}^{\infty} |\widehat{\varphi}(n)| = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty,$$

so $\widehat{\varphi} \in \ell^1(\mathbb{Z})$. Therefore the inversion formula gives

$$\varphi(x) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n) e^{2\pi i n x}$$

pointwise. (In fact one can get uniform convergence on compact sets.) □

Remark. Formally consider $\mu = \sum_{n=-\infty}^{\infty} \delta_n$, where δ_n is the Dirac delta at n . Then

$$\langle f, \mu \rangle = \int_{-\infty}^{\infty} f(x) d\mu(x) = \sum_{n=-\infty}^{\infty} f(n).$$

A similar computation shows that

$$\langle \widehat{f}, \mu \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

These series are equal by Poisson summation, so if the Parseval identity holds, then one gets

$$\langle f, \check{\mu} \rangle = \langle \widehat{f}, \mu \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n) = \sum_{n=-\infty}^{\infty} f(n) = \langle f, \mu \rangle.$$

Thus Poisson summation says that $\mu = \check{\mu}$.

22.2 Wiener's Lemma

Remark. Let $C(\mathbb{T}) = \{f \text{ continuous on } \mathbb{T}\}$, which is closed under products and

$$\|fg\|_{\infty} \leq \|f\|_{\infty} \|g\|_{\infty}.$$

In particular, $C(\mathbb{T})$ is a commutative Banach algebra. Also note that if $f \in C(\mathbb{T})$ and $f(x) \neq 0$ for every x , then $1/f \in C(\mathbb{T})$. We say that $C(\mathbb{T})$ is *inverse-closed*.

Recall that the Wiener (Fourier) algebra is

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f} \in \ell^1(\mathbb{Z})\} = \{\widehat{c} : c = (c_n) \in \ell^1(\mathbb{Z})\}.$$

The equality holds by the formulas $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi i n x}$ and $\check{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$. Note that $A(\mathbb{T}) \subseteq C(\mathbb{T})$ is dense in the uniform norm.

On the other hand, if we define

$$\|f\|_{A(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|,$$

then for any $f, g \in A(\mathbb{T})$, we have

$$fg = (\widehat{fg})^{\vee} = (\widehat{f} * \widehat{g})^{\vee}.$$

Since $\widehat{f}, \widehat{g} \in \ell^1(\mathbb{Z})$, we have $\widehat{f} * \widehat{g} \in \ell^1(\mathbb{Z})$, so $fg \in A(\mathbb{T})$. Moreover,

$$\|fg\|_{A(\mathbb{T})} = \|\widehat{f} * \widehat{g}\|_1 \leq \|\widehat{f}\|_1 \|\widehat{g}\|_1 = \|f\|_{A(\mathbb{T})} \|g\|_{A(\mathbb{T})}.$$

Thus we see that $A(\mathbb{T})$ is also a commutative Banach algebra with respect to pointwise products.

If $A(\mathbb{T})$ is inverse-closed? In other words, if $f \in A(\mathbb{T})$ and $f(x) \neq 0$ for all x , must $1/f \in A(\mathbb{T})$?

Lemma 22.1. *If $P(\xi) = \sum_{k=-N}^N a_k e^{2\pi i k \xi}$, then $\|P\|_{A(\mathbb{T})} \leq (2N+1)^{1/2} \|P\|_{\infty}$.*

Proof. By Cauchy-Schwarz, we have

$$\|P\|_{A(\mathbb{T})} = \sum_{k=-N}^N |a_k| \cdot 1 \leq (2N+1)^{1/2} \left(\sum_{k=-N}^N |a_k|^2 \right)^{1/2} = (2N+1)^{1/2} \|P\|_2$$

where the second equality is by Plancherel's theorem. The result follows since $\|P\|_2 \leq \|P\|_{\infty}$ on $\mathbb{T}$. $\square$

Lemma 22.2 (Wiener's lemma). *If $g \in \ell^1(\mathbb{Z})$ and $\widehat{g}(\xi) \neq 0$ for any ξ , then there is $h \in \ell^1(\mathbb{Z})$ such that*

$$\widehat{h}(\xi) = \frac{1}{\widehat{g}(\xi)}, \quad \xi \in \mathbb{T}.$$

Equivalently, if $G \in A(\mathbb{T})$ and $G(\xi) \neq 0$ for any ξ , then $1/G \in A(\mathbb{T})$.

Proof. Assume $g \in \ell^1(\mathbb{Z})$ and $\widehat{g}(\xi) \neq 0$ for any ξ . Let $G = \widehat{g}$. We prove the result in two steps:

1. Assume $0 \leq G \leq 1$. Then since G is nonzero,

$$d = \inf_{\xi \in \mathbb{T}} G(\xi) > 0.$$

Let $H = 1 - G = \widehat{\delta} - \widehat{g}$, where δ is the delta sequence. Note that

$$\widehat{\delta}(\xi) = \sum_{n=-\infty}^{\infty} \delta_n e^{2\pi i n \xi} = \delta_0 e^{2\pi i 0 \xi} = 1.$$

Then $H = \widehat{h} - \widehat{g} = (\delta - g)^\wedge \in A(\mathbb{T})$. So if we let $h = \delta - g \in \ell^1(\mathbb{Z})$, then $\widehat{h} = H$, and

$$\|H\|_\infty = \|1 - G\|_\infty = 1 - d < 1.$$

Hence $\sum_{n=0}^{\infty} H(\xi)^n$ converges, and $\sum_{n=0}^{\infty} H^n = 1/(1 - H(\xi)) = 1/G(\xi) \in C(\mathbb{T})$.

However, we need convergence in $A(\mathbb{T})$. Fix $0 < \epsilon < d/2$. Let $p = h\chi_{[-N, N]}$ for N large enough so that $\|p - h\|_1 < \epsilon$. Let $P = \widehat{p} = \sum_{n=-N}^N h(n)e^{2\pi i n \xi}$. Let $r = p - h$ and $R = P - H$. Then

$$\begin{aligned} \|P\|_\infty &= \|H + R\|_\infty \leq \|H\|_\infty + \|R\|_\infty \leq (1 - d) + \|P - H\|_\infty \\ &\leq (1 - d) + \|\widehat{p} - \widehat{h}\|_\infty \leq (1 - d) + \|p - h\|_1 = 1 - d + \epsilon. \end{aligned}$$

Now we can compute that

$$\|H^n\|_{A(\mathbb{T})} = \|(P - R)^n\|_{A(\mathbb{T})} = \left\| \sum_{j=0}^n \binom{n}{j} P^j (-R)^{n-j} \right\|_{A(\mathbb{T})}$$

where we can use the binomial theorem since $A(\mathbb{T})$ is a commutative Banach algebra. Then

$$\|H^n\|_{A(\mathbb{T})} \leq \sum_{j=0}^n \binom{n}{j} \|P\|_{A(\mathbb{T})}^j \|R\|_{A(\mathbb{T})}^{n-j} \leq \sum_{j=0}^n \binom{n}{j} (2Nj + 1)^{1/2} \|P^j\|_\infty \epsilon^{n-j},$$

where the second equality is by Lemma 22.1 and $\|R\|_{A(\mathbb{T})} = \|p - h\|_1 < \epsilon$. Thus

$$\|H^n\|_{A(\mathbb{T})} \leq (2Nn + 1)^{1/2} \sum_{j=1}^n \binom{n}{j} \|P\|_\infty^j \epsilon^{n-j} = (2Nn + 1)^{1/2} (\|P\|_\infty + \epsilon)^n,$$

where $\|P^j\|_\infty \leq \|P\|_\infty^j$ by submultiplicativity. Since $\|P\|_\infty \leq 1 - d + \epsilon$,

$$\|H^n\|_{A(\mathbb{T})} \leq (2Nn + 1)^{1/2} (1 - d + 2\epsilon)^n,$$

where $1 - d + 2\epsilon < 1$. So $\sum_{n=0}^{\infty} \|H^n\|_{A(\mathbb{T})} < \infty$, so $\sum_{n=0}^{\infty} H^n$ converges in $A(\mathbb{T})$ to $1/(1 - H) = 1/G$.

One can reduce the general case to this first case, see the course notes. $\square$

22.3 Distributions

Remark. Let $1 \leq p < \infty$. Note that $L^p(\mathbb{R})^* \cong L^{p'}(\mathbb{R})$, where $L^p(\mathbb{R})^*$ is the dual space of $L^p(\mathbb{R})$, i.e. the space of bounded linear functions on $L^p(\mathbb{R})$. If we fix $g \in L^{p'}(\mathbb{R})$, then we can define

$$\mu_g(f) = \langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx,$$

which is linear in f and antilinear in g (this is known as a *sesquilinear form*).

Recall that $X^* = \{\text{bounded linear functionals } \mu \text{ on } X\}$, which is equivalent to continuous linear functionals when X is a Banach space. We consider the following space of distributions:

Definition 22.1. Define the following:

- $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$, the space of *distributions*.
- $\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$, the space of *tempered distributions*.
- $\mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$, the space of *compactly supported distributions*.

Example 22.1.1. If $f \in C_c^\infty(\mathbb{R})$, then define $\langle f, \delta \rangle = f(0)$. This is the Dirac delta as a distribution.

Remark. Recall that the Schwartz space is

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R})\}.$$

This is *not* a Banach space (it does not have a norm). But we can define seminorms

$$\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty, \quad m, n \geq 0.$$

There is no way to combine these seminorms into a single norm.¹ However, one can define a metric:

$$d(f, g) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \frac{\rho_{m,n}(f - g)}{1 + \rho_{m,n}(f - g)}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Note that convergence with respect to d is the same as convergence in $\rho_{m,n}$ for every m, n :

$$\lim_{k \rightarrow \infty} \rho_{m,n}(f - f_k) = 0.$$

Definition 22.2. We say that $f_k \rightarrow f$ in $\mathcal{S}(\mathbb{R})$ if for every $m, n \geq 0$,

$$\rho_{m,n}(f - f_k) = \|x^m f^{(n)}(x) - x^m (f_k)^{(n)}(x)\|_\infty \longrightarrow 0$$

Therefore, a linear functional $\mu : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ is *continuous* if $f_k \rightarrow f$ in $\mathcal{S}(\mathbb{R})$ implies $\langle f_k, \mu \rangle \rightarrow \langle f, \mu \rangle$.

Remark. Since convergence in $\mathcal{S}(\mathbb{R})$ is a very strong condition, we expect that it is easy for μ to satisfy the above condition. This intuitively explains why $\mathcal{S}'(\mathbb{R})$ is so large.

Example 22.2.1. Assume that $f_k \rightarrow f$ in $\mathcal{S}(\mathbb{R})$. Then $\langle f_k, \delta \rangle = f_k(0)$, so

$$|\langle f - f_k, \delta \rangle| = |f(0) - f_k(0)| \leq \rho_{0,0}(f - f_k) \longrightarrow 0.$$

Thus we see that $\delta \in \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$.

¹Compare this with $C_b^1(\mathbb{R}) = \{f : \|f\|_\infty, \|f'\|_\infty < \infty\}$, where one can define a norm $\|f\|_{C_b^1} = \|f\|_\infty + \|f'\|_\infty$.

Lecture 23

Nov. 20 — Distributions

23.1 Convergence with Families of Seminorms

Example 23.0.1. Like $\mathcal{S}(\mathbb{R})$, many other spaces are defined by a family of seminorms. Recall that

$$L^1_{\text{loc}}(\mathbb{R}) = \{\text{measurable } f : f \text{ is integrable on compact } K \subseteq \mathbb{R}\}.$$

If we define $\rho_K(f) = \|f\chi_K\|_1$, then $f \in L^1_{\text{loc}}(\mathbb{R})$ if and only if $\rho_K(f) < \infty$. In fact, it is enough to consider $\rho_N(f) = \|f\chi_{[-N,N]}\|_1$. Given such a countable family of seminorms, one can always define a metric

$$d(f, g) = \sum_{N=1}^{\infty} 2^{-N} \frac{\rho_N(f - g)}{1 + \rho_N(f - g)}$$

which satisfies $d(f, f_k) \rightarrow 0$ if and only if $\rho_N(f - f_k) \rightarrow 0$ for every N .

Remark. Suppose $\|\cdot\|$ is a norm on $L^1_{\text{loc}}(\mathbb{R})$, and consider $\chi_{[k,k+1]}$. Then $\rho_N(\chi_{[k,k+1]}) \rightarrow 0$, so

$$c_k \chi_{[k,k+1]} \rightarrow 0$$

in $L^1_{\text{loc}}(\mathbb{R})$ for every choice of c_k . But then $\|c_k \chi_{[k,k+1]}\| \rightarrow 0$, a contradiction for $c_k = 1/\|\chi_{[k,k+1]}\|$.

Example 23.0.2. For $C^\infty(\mathbb{R})$, we can take $\rho_{N,n}(f) = \|f^{(n)}\chi_{[-N,N]}\|_\infty$. Then

$$C^\infty(\mathbb{R})^* = \{\text{continuous linear functionals } \mu : C^\infty(\mathbb{R}) \rightarrow \mathbb{C}\} = \mathcal{E}'(\mathbb{R})$$

is the space of compactly supported distributions, where μ is continuous if whenever $f_k \rightarrow f$ in $C^\infty(\mathbb{R})$, we have $\mu(f_k) = \langle \mu, f_k \rangle \rightarrow \langle \mu, f \rangle$.

Remark. For $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$, it is better to think of $C_c^\infty(\mathbb{R}) = \bigcup_{N=1}^{\infty} C_c^\infty([-N, N])$.

Definition 23.1. Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space X .

- (a) $f_k \rightarrow f$ in X means that $\rho_\alpha(f - f_k) \rightarrow 0$ for every $\alpha \in J$.
- (b) X is *Hausdorff* if $\rho_\alpha(f) = 0$ for every $f \in X$ if and only if $f = 0$.

Definition 23.2. Let $\{\rho_n\}_{n \in \mathbb{N}}$ be a countable, Hausdorff family of seminorms on X . If the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f - g)}{1 + \rho_n(f - g)}$$

is complete, then we call X a *Fréchet space*.

Exercise 23.1. $\mathcal{S}(\mathbb{R})$ and $C^\infty(\mathbb{R})$ are Fréchet spaces (but $C_c^\infty(\mathbb{R})$ is not).

Remark. Recall that if $\mu : X \rightarrow \mathbb{C}$ is a linear functional on a normed space X , then μ is continuous if and only if μ is bounded (i.e. there exists $C > 0$ such that $|\langle f, \mu \rangle| \leq C\|f\|$ for every $f \in X$).

Theorem 23.1. Let X be a Fréchet space with seminorms $\{\rho_n\}_{n \in \mathbb{N}}$ and $\mu : X \rightarrow \mathbb{C}$ a linear functional. Then the following are equivalent:

1. μ is continuous, i.e. $f_k \rightarrow f$ implies $\langle f_k, \mu \rangle \rightarrow \langle f, \mu \rangle$ as $k \rightarrow \infty$.
2. $f_k \rightarrow 0$ in X implies $\langle f_k, \mu \rangle \rightarrow 0$.
3. There exists $C > 0$ and $N \in \mathbb{N}$ (depending on μ) such that $|\langle f, \mu \rangle| \leq C \sum_{n=1}^N \rho_n(f)$.

Proof. (1 $\Leftrightarrow$ 2) This follows from linearity.

(3 $\Rightarrow$ 2) Suppose $f_k \rightarrow 0$, so $\rho_n(f_k) \rightarrow 0$ for every n . So

$$|\langle f_k, \mu \rangle| \leq C \sum_{n=1}^N \rho_n(f_k) \xrightarrow{k \rightarrow \infty} 0,$$

since each of the finitely many terms converge to 0.

(2 $\Rightarrow$ 3) We prove the contrapositive. Suppose (3) is false. So for every $C = N = k \in \mathbb{N}$, there exists $f_k \in X$ for which $|\langle f_k, \mu \rangle| > k \sum_{n=1}^k \rho_n(f_k)$. Set $\varphi_k = f_k / |\langle f_k, \mu \rangle|$. Then

$$\sum_{n=1}^k \rho_n(\varphi_k) = \frac{1}{|\langle f_k, \mu \rangle|} \sum_{n=1}^k \rho_n(f_k) < \frac{1}{|\langle f_k, \mu \rangle|} \cdot \frac{|\langle f_k, \mu \rangle|}{k} = \frac{1}{k} \xrightarrow{k \rightarrow \infty} 0.$$

Thus $\rho_n(\varphi_k) \rightarrow 0$ as $k \rightarrow \infty$ for every $n \in \mathbb{N}$. So $\varphi_k \rightarrow 0$ in X . But $\langle \varphi_k, \mu \rangle = 1 \not\rightarrow 0$. $\square$

Example 23.2.1. What does the topology look like given a family of seminorms, i.e. what are the open sets? Consider seminorms $\rho_1(x_1, x_2) = |x_1|$ and $\rho_2(x_1, x_2) = |x_2|$. Consider strips

$$B_r^i(x_1, x_2) = \{(y_1, y_2) : |y_i - x_i| < r\}$$

for each seminorm ρ_i . Then the open balls are of the form

$$B_r^1(x_1, x_2) \cap B_r^2(x_1, x_2).$$

A basis in the general case involves intersections of finitely many strips (as in the product topology).

23.2 Distributional Derivatives

Example 23.2.2. If f, g are differentiable and decay at ∞ (think $f, g \in \mathcal{S}(\mathbb{R})$), then

$$\langle f, g' \rangle = \int_{-\infty}^{\infty} f(x) \overline{g'(x)} dx = f(x) \overline{g(x)} \Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \overline{g(x)} dx = -\langle f', g \rangle,$$

by integration by parts, where $f(x) \overline{g(x)} \Big|_{x=-\infty}^{\infty} = 0$ by the decay of f, g . Now define $\delta' : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ by

$$\langle f, \delta' \rangle := -\langle f', \delta \rangle = -f'(0).$$

for $f \in \mathcal{S}(\mathbb{R})$. Then δ' is a linear functional, and

$$|\langle f, \delta' \rangle| = |-f'(0)| \leq \|f'\|_\infty = \|x^0 f^{(1)}(x)\|_\infty = \rho_{0,1}(f) \leq 1 \sum_{m=0}^0 \sum_{n=0}^1 \rho_{m,n}(f).$$

Hence δ' is bounded on $\mathcal{S}(\mathbb{R})$, so $\delta' \in \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$.

Proposition 23.1. Fix $\mu \in \mathcal{S}'(\mathbb{R})$, and define $\mu' : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ by

$$\langle f, \mu' \rangle = -\langle f', \mu \rangle, \quad f \in \mathcal{S}(\mathbb{R}).$$

Then $\mu' \in \mathcal{S}'(\mathbb{R})$. In particular, every tempered distribution has a (distributional) derivative.

Proof. Since μ is bounded, there are $C, N > 0$ such that $|\langle f, \mu \rangle| \leq C \sum_{m=0}^M \sum_{n=0}^N \rho_{m,n}(f)$. Then

$$|\langle f, \mu' \rangle| = |\langle f', \mu \rangle| \leq C \sum_{m=0}^M \sum_{n=0}^N \rho_{m,n}(f') = C \sum_{m=0}^M \sum_{n=0}^N \|x^m (f')^{(n)}\|_\infty = C \sum_{m=0}^M \sum_{n=0}^N \|x^m f^{n+1}\|_\infty,$$

so μ' is bounded. Thus $\mu' \in \mathcal{S}'(\mathbb{R})$ also. □

Example 23.2.3. Consider the Heaviside function $H = \chi_{[0,\infty)}$. We can define

$$\langle f, H \rangle = \int_{-\infty}^{\infty} f(x) \overline{H(x)} dx = \int_0^{\infty} f(x) dx.$$

Then we can check that

$$\begin{aligned} |\langle f, H \rangle| &\leq \int_0^{\infty} |f(x)| dx \leq \int_0^1 |f(x)| dx + \int_1^{\infty} \frac{|f(x)|}{x^2} x^2 dx \\ &\leq \|f\|_\infty + \|x^2 f(x)\|_\infty \int_1^{\infty} \frac{1}{x^2} dx \leq C \sum_{m=0}^2 \sum_{n=0}^0 \|x^m f^{(n)}\|_\infty. \end{aligned}$$

Thus H is bounded, so $H \in \mathcal{S}'(\mathbb{R})$. So H has a distributional derivative $H' : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$, where

$$\langle f, H' \rangle := -\langle f', H \rangle = -\int_0^{\infty} f'(x) dx = -(f(\infty) - f(0)) = f(0) = \langle f, \delta \rangle.$$

In particular, we see that $H' = \delta$ as tempered distributions.

Remark. Note that $L_{\text{loc}}^1(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$ and $L_{\text{loc}}^1(\mathbb{R}) + \text{polynomial growth} \subseteq \mathcal{S}'(\mathbb{R})$, so each of the functions in the latter has distributional derivatives.

Exercise 23.2. If g is smooth, then g' (as a tempered distribution) is the usual derivative.

Remark. To avoid confusion, we sometimes write $D\mu$ for the distributional derivative and g' for the pointwise a.e. derivative.

Lecture 24

Nov. 25 — Distributions, Part 2

24.1 Space of Distributions

Remark. Let $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$. The topology on this space is *not* given by a family of seminorms. Instead, we write $C_c^\infty(\mathbb{R}) = \bigcup_{K \text{ compact}} C^\infty(K)$. Note that $C^\infty(K)$ is a Fréchet space with seminorms

$$\rho_{K,n}(f) = \|f^{(n)}(x)\chi_K(x)\|_\infty$$

for $f \in C^\infty(K) = \{f \in C^\infty(\mathbb{R}) : \text{supp}(f) \subseteq K\}$.

Theorem 24.1. $f_k \rightarrow f$ in $C_c^\infty(\mathbb{R})$ if and only if there exists a compact set $K \subseteq \mathbb{R}$ such that $f_k \in C^\infty(K)$ and $f_k \rightarrow f$ in $C^\infty(K)$.

Theorem 24.2. Given a linear function $\mu : C_c^\infty(\mathbb{R}) \rightarrow \mathbb{C}$, the following are equivalent:

1. μ is continuous;
2. $\mu|_{C^\infty(K)}$ is continuous for each compact K ;
3. for all compact $K \subseteq \mathbb{R}$, there exists $C_K \geq 0$, $N_K \geq 0$ such that

$$|\langle f, \mu \rangle| \leq C_K \sum_{n=0}^{N_K} \|f^{(n)}(x)\chi_K(x)\|_\infty, \quad \text{for } f \in C^\infty(K).$$

Definition 24.1. If there exists a single N that can be used for each compact set K in Theorem 24.2(3), then we say that μ has *finite order*. The *order* of μ is the smallest such N .

Example 24.1.1. We have the following:

1. The order of $\delta^{(j)}$ is j .
2. Consider the δ -train $\mu = \sum_{n=-\infty}^{\infty} \delta_n$, so $\langle f, \mu \rangle = \sum_{n=-\infty}^{\infty} f(n)$. Note that this sum is finite since $f \in C^\infty(K)$ for some compact K . In fact, $|\langle f, \mu \rangle| \leq C_K \|f\|_\infty$, so the order is 0.
3. Let $\nu = \sum_{n=1}^{\infty} \delta_n^{(n)}$, so $\langle f, \nu \rangle = \sum_{n=1}^{\infty} (-1)^n f^{(n)}(n)$. One can check that ν has infinite order.

Remark. As sets, we have $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$. What about their duals?

Theorem 24.3. We have the following:

1. $f_k \rightarrow f$ in $C_c^\infty(\mathbb{R})$ implies $f_k \rightarrow f$ in $\mathcal{S}(\mathbb{R})$.
2. $f_k \rightarrow f$ in $\mathcal{S}(\mathbb{R})$ implies $f_k \rightarrow f$ in $C^\infty(\mathbb{R})$.

3. $\mu \in \mathcal{S}'(\mathbb{R})$ implies $\mu|_{C_c^\infty(\mathbb{R})} \in \mathcal{D}'(\mathbb{R})$.

4. $\mu \in \mathcal{E}'(\mathbb{R})$ implies $\mu|_{\mathcal{S}(\mathbb{R})} \in \mathcal{S}'(\mathbb{R})$.

In particular, we have the containments $\mathcal{D}'(\mathbb{R}) \supseteq \mathcal{S}'(\mathbb{R}) \supseteq \mathcal{E}'(\mathbb{R})$.

24.2 Functions as Distributions

Theorem 24.4. $L_{\text{loc}}^1(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$, where the embedding is given by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx, \quad f \in C_c^\infty(\mathbb{R}).$$

Proof. If $g \in L_{\text{loc}}^1(\mathbb{R})$ and $f \in C_c^\infty(K)$, then

$$\langle f, g \rangle \leq \int_K |f(x)g(x)| dx \leq \|f\|_\infty \|g\chi_K\|_1.$$

Thus we see that g defines a continuous linear functional on $C_c^\infty(\mathbb{R})$. □

Example 24.1.2. Note that $\delta \in \mathcal{D}'(\mathbb{R})$, but it cannot be identified with a function in $L_{\text{loc}}^1(\mathbb{R})$.

Example 24.1.3. Consider the space

$$\begin{aligned} M_b(\mathbb{R}) &= \{\text{bounded Radon measures on } \mathbb{R}\} \\ &= \{\text{bounded locally finite Borel measures on } \mathbb{R}\}. \end{aligned}$$

If $\mu \in M_b(\mathbb{R})$ and $f \in C_c^\infty(K)$, then we can define a linear functional by

$$\langle f, \mu \rangle = \int f(x) d\mu(x).$$

Then we have that

$$|\langle f, \mu \rangle| \leq \int_K |f(x)| d|\mu|(x) \leq \|f\|_\infty |\mu|(K),$$

which is finite as μ is a bounded Radon measure.

Example 24.1.4. We have $1/x \notin L_{\text{loc}}^1(\mathbb{R})$, but we can define $\text{pv}(1/x) : C_c^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ by

$$\langle f, \text{pv}(1/x) \rangle = \lim_{T \rightarrow \infty} \int_{1/T \leq |x| \leq T} \frac{f(x)}{x} dx, \quad f \in C_c^\infty(K).$$

One can show that the above limit exists, so $\text{pv}(1/x) \in \mathcal{D}'(\mathbb{R})$.

24.3 Operations on Distributions

Remark. Suppose $g \in L_{\text{loc}}^1(\mathbb{R})$, and set $T_a g(x) = g(x - a)$. If $f \in C_c^\infty(\mathbb{R})$, then

$$\langle f, T_a g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x - a)} dx = \int_{-\infty}^{\infty} f(x + a) \overline{g(x)} dx = \langle T_{-a} f, g \rangle.$$

Definition 24.2. If $\mu \in \mathcal{D}'(\mathbb{R})$, then define $T_a\mu : C_c^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ by $\langle f, T_a\mu \rangle := \langle T_{-a}f, \mu \rangle$ for $f \in C_c^\infty(\mathbb{R})$.

Remark. Note that for $f \in C^\infty(K)$, we have

$$|\langle f, T_a\mu \rangle| = |\langle T_{-a}f, \mu \rangle| \leq C_{K+a} \sum_{n=0}^{N_{K+a}} \|T_{-a}f\|_\infty \leq C_{K+a} \sum_{n=0}^{N_{K+a}} \|f\|_\infty$$

since $T_{-a}f \in C^\infty(K+a)$. Thus $T_a\mu$ is a continuous linear functional on $C_c^\infty(\mathbb{R})$,

Remark. One can similarly define other operations on distributions, e.g. dilation, modulation, etc. For example, the *involution* $\tilde{g}(x) = \overline{g(-x)}$ satisfies

$$\langle f, \tilde{g} \rangle = \int f(x) \overline{\tilde{g}(-x)} dx = \int \overline{f(x)g(-x)} dx = \int \overline{f(-x)g(x)} dx = \int \tilde{f}(x) \overline{g(x)} dx = \overline{\langle \tilde{f}, g \rangle}.$$

Thus if $\mu \in \mathcal{D}'(\mathbb{R})$, then we can define $\langle f, \tilde{\mu} \rangle := \overline{\langle \tilde{f}, \mu \rangle}$ for $f \in C_c^\infty(\mathbb{R})$.

Example 24.2.1. Consider a translation of δ :

$$\langle f, T_a\delta \rangle := \langle T_{-a}f, \delta \rangle = T_{-a}f(0) = f(a).$$

But sometimes by abuse of notation we may write

$$\langle f, T_a\delta \rangle = \int f(x) \delta(x-a) dx = \int f(x+a) \delta(x) dx = f(a).$$

This should be understood in the distributional sense, the above integral is *not* a Lebesgue integral.

24.4 Products and Convolution of Distributions

Remark. Suppose $g \in L_{\text{loc}}^1(\mathbb{R})$, $\theta \in C^\infty(\mathbb{R})$. Then

$$\langle f, \theta g \rangle = \int f(x) \overline{\theta(x)g(x)} dx = \int (f(x)\overline{\theta(x)}) \overline{g(x)} dx = \langle f\bar{\theta}, g \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Note that $f\bar{\theta} \in C_c^\infty(\mathbb{R})$, so the above definition makes sense.

Definition 24.3. If $\mu \in \mathcal{D}'(\mathbb{R})$ and $\theta \in C^\infty(\mathbb{R})$, then define $\theta\mu : C_c^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ by $\langle f, \theta\mu \rangle := \langle f\bar{\theta}, \mu \rangle$.

Remark. We can check that for $f \in C^\infty(K)$ (so $f\bar{\theta} \in C^\infty(K)$),

$$\begin{aligned} |\langle f, \theta\mu \rangle| &= |\langle f\bar{\theta}, \mu \rangle| \leq C_K \sum_{n=0}^{N_K} \|(f\bar{\theta})^{(n)}\|_\infty \leq C_K \sum_{n=0}^{N_K} \sum_{j=0}^n \binom{n}{j} \|f^{(j)}\overline{\theta^{(n-j)}}\|_\infty \\ &= C_K \sum_{n=0}^{N_K} \sum_{j=0}^n \binom{n}{j} \|\theta^{(n-j)}\|_\infty \|f^{(j)}\|_\infty. \end{aligned}$$

Thus we see that $\theta\mu \in \mathcal{D}'(\mathbb{R})$.

Remark. Now consider $g \in L_{\text{loc}}^1(\mathbb{R})$, $f \in C_c^\infty(\mathbb{R})$. Then we have

$$(f * g)(x) = \int f(x-y)g(y) dy = \int \overline{\tilde{f}(y-x)}g(y) dy = \overline{\langle T_x\tilde{f}, g \rangle}.$$

Definition 24.4. The *convolution* of $\mu \in \mathcal{D}'(\mathbb{R})$ with $f \in C_c^\infty(\mathbb{R})$ is

$$(f * \mu)(x) := \overline{\langle T_x \widetilde{f}, \mu \rangle}, \quad x \in \mathbb{R}.$$

Theorem 24.5. If $\mu \in \mathcal{D}'(\mathbb{R})$ and $f \in C_c^\infty(\mathbb{R})$, then:

1. Convolution commutes with translation, i.e. $T_a(f * \mu) = (T_a f) * \mu = f * (T_a \mu)$.
2. $f * \mu \in C^\infty(\mathbb{R})$ and $(f * \mu)' = f' * \mu$ (and also $(f * \mu)' = f * D\mu$).

Remark. Suppose f, g, h are “nice” functions. Then

$$\begin{aligned} \langle f, g * h \rangle &= \int f(x) \overline{(g * h)(x)} dx = \iint f(x) \overline{g(x-y)h(y)} dy dx \\ &= \iint (f(x) \widetilde{g}(y-x) dx) \overline{h(y)} dy = \int (f * \widetilde{g})(y) \overline{h(y)} dy = \langle f * \widetilde{g}, h \rangle. \end{aligned}$$

Theorem 24.6. If $\mu \in \mathcal{D}'(\mathbb{R})$ and $f, g \in C_c^\infty(\mathbb{R})$, then $\langle f, g * \mu \rangle = \langle f * \widetilde{g}, \mu \rangle$.

Lecture 25

Dec. 2 — Distributions, Part 3

25.1 More on Convolutions of Distributions

Remark. Recall that if $\nu \in \mathcal{D}'(\mathbb{R})$ and $f, g \in C_c^\infty(\mathbb{R})$, then we define

$$\langle f, g * \nu \rangle := \langle f * \tilde{g}, \nu \rangle.$$

We want to make a similar definition for $\mu, \nu \in \mathcal{D}'(\mathbb{R})$:

$$\langle f, \mu * \nu \rangle := \langle f * \tilde{\mu}, \nu \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Note that we only know $f * \mu \in C^\infty(\mathbb{R})$, not necessarily $C_c^\infty(\mathbb{R})$. We know

$$\text{supp}(\mu) = \bigcap \{\text{closed sets } F \subseteq \mathbb{R} : \mu = 0 \text{ off } F\}.$$

Theorem 25.1. $\mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^* = \{\mu \in \mathcal{D}'(\mathbb{R}) : \mu \text{ is compactly supported}\}$.

Definition 25.1. If $\mu, \nu \in \mathcal{D}'(\mathbb{R})$ and at least one is compactly supported, then we can define

$$\langle f, \mu * \nu \rangle := \langle f * \tilde{\mu}, \nu \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Exercise 25.1. Show that if at least one of ν or μ is compactly supported, then

$$f * (\mu * \nu) = (f * \mu) * \nu, \quad f \in C_c^\infty(\mathbb{R}).$$

Exercise 25.2. Show that $(H * \delta') * 1 \neq H * (\delta' * 1)$, where H is the Heaviside function, δ is the Dirac delta, and 1 is the constant function.

25.2 More on Distributional Derivatives

Lemma 25.1. Let $g \in C(\mathbb{R})$ such that $Dg \in C(\mathbb{R})$. Then $g \in C^1(\mathbb{R})$ and $Dg = g'$.

Proof. We give the idea of the proof. Let $h \in \mathbb{R}$ and suppose $f \in C_c^\infty(\mathbb{R})$. Then

$$\int_0^h (\tilde{f} * g)'(t) dt = (\tilde{f} * g)(h) - (\tilde{f} * g)(0)$$

by the fundamental theorem of calculus. Check as an exercise that $(\tilde{f} * g)'(t) = (\tilde{f} * Dg)(t)$, then

$$\int \overline{f(x)} g(x) dx = \overline{\langle f, g \rangle} = (\tilde{f} * g)(h) - \int_0^h (\tilde{f} * Dg)(t) dt.$$

One can continue to show that Dg and g' agree. □

Exercise 25.3. Show that if $\mu \in \mathcal{D}'(\mathbb{R})$ and $f \in C_c^\infty(\mathbb{R})$, then

$$\overline{\langle f, \mu \rangle} = (\tilde{f} * \mu)(t) - \int_0^h (\tilde{f} * D\mu)(t) dt.$$

Theorem 25.2. If $\mu \in \mathcal{D}'(\mathbb{R})$, then $D\mu \in C(\mathbb{R})$ (i.e. there exists $g \in C(\mathbb{R})$ such that $\langle f, g \rangle = \langle f, D\mu \rangle = -\langle f', \mu \rangle$) if and only if $\mu \in C^1(\mathbb{R})$.

Theorem 25.3. Fix $\mu \in \mathcal{D}'(\mathbb{R})$. Then for each compact $K \subseteq \mathbb{R}$, there exists $g \in C(\mathbb{R})$ and $k \geq 0$ such that $\langle f, \mu \rangle = \langle f, D^k g \rangle$ for all $f \in C^\infty(K)$.

Remark. Theorem 25.3 says that in some sense, distributions are exactly the derivatives of functions.

25.3 Distributional Fourier Transform

Remark. Recall that the Fourier transform is unitary on $L^2(\mathbb{R})$, i.e.

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = \int_{-\infty}^{\infty} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi = \langle \widehat{f}, \widehat{g} \rangle.$$

Also recall that if $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$ also (this is not the case with $C_c^\infty(\mathbb{R})$).

Definition 25.2. The *Fourier transform* of a tempered distribution $\mu \in \mathcal{S}'(\mathbb{R})$ is implicitly defined by

$$\langle \widehat{f}, \widehat{\mu} \rangle := \langle f, \mu \rangle, \quad f \in \mathcal{S}(\mathbb{R}).$$

This defines $\widehat{\mu}$ on all of $\mathcal{S}(\mathbb{R})$ since $\widehat{f}$ can be any element of $\mathcal{S}(\mathbb{R})$. More explicitly,

$$\langle f, \widehat{\mu} \rangle = \langle \check{f}, \mu \rangle, \quad f \in \mathcal{S}(\mathbb{R}).$$

Lemma 25.2. If $\mu \in \mathcal{S}'(\mathbb{R})$, then $\widehat{\mu} \in \mathcal{S}'(\mathbb{R})$.

Proof. Since $\mu \in \mathcal{S}'(\mathbb{R})$, there exist $C, M, N \geq 0$ such that

$$|\langle f, \mu \rangle| \leq C \sum_{m=0}^M \sum_{n=0}^N \|x^m f^{(n)}\|_\infty.$$

Then we can compute

$$|\langle f, \widehat{\mu} \rangle| = |\langle \check{f}, \mu \rangle| \leq C \sum_{m=0}^M \sum_{n=0}^N \|x^m \check{f}^{(n)}\|_\infty.$$

For $m = n = 0$, we have $\|\check{f}\|_\infty \leq \|f\|_1 \leq C'(\|f\|_\infty + \|x^2 f(x)\|_\infty)$ (split into $|x| \leq 1$ and $|x| > 1$). We can apply a similar argument to the other m, n to bound the remaining terms. $\square$

Example 25.2.1. We compute $\widehat{\delta}$. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\langle \widehat{f}, \widehat{\delta} \rangle = \langle f, \delta \rangle = f(0) = (\widehat{f})^\vee(0) = \int_{-\infty}^{\infty} \widehat{f}(\xi) d\xi = \langle \widehat{f}, 1 \rangle.$$

This is true for every $f \in \mathcal{S}(\mathbb{R})$, hence $\widehat{\delta} = 1$.

Example 25.2.2. We compute $\widehat{1}$. For $f \in \mathcal{S}(\mathbb{R})$, we have

$$\langle \widehat{f}, \widehat{1} \rangle = \langle f, 1 \rangle = \int f(x) dx = \widehat{f}(0) = \langle \widehat{f}, \delta \rangle.$$

Thus we see that $\widehat{1} = \delta$.

Example 25.2.3. Formally, we can write

$$\widehat{\delta}(\xi) = \int_{-\infty}^{\infty} \delta(x) e^{-2\pi i \xi x} dx = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} d\delta(x) = e^{-2\pi i \xi \cdot 0} = 1.$$

This somewhat makes sense, with some hand-waving. On the other hand

$$\widehat{1}(\xi) = \int_{-\infty}^{\infty} 1 \cdot e^{-2\pi i \xi x} dx = \delta(\xi)$$

does not really make sense at all.

Exercise 25.4. Show that $\mu \mapsto \widehat{\mu}$ is a continuous bijection $\mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$.

Exercise 25.5. Show that if $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R})$, then the usual Fourier transform agrees with the distributional Fourier transform (if we interpret the usual $\widehat{f} \in L^{p'}(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$).

Remark. If $2 < p \leq \infty$, then we still have $L^p(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$, so every $f \in L^p(\mathbb{R})$ has a distributional Fourier transform $\widehat{f} \in \mathcal{S}'(\mathbb{R})$, but in general it cannot be identified with a function.

Remark. $\mathcal{FL}^\infty(\mathbb{R})$ is called the space of *pseudo-measures*.

Exercise 25.6. Show that if $\mu \in \mathcal{S}'(\mathbb{R})$ and $\theta \in \mathcal{S}(\mathbb{R})$, then

$$(\theta\mu)^\wedge = \widehat{\theta} * \widehat{\mu} \quad \text{and} \quad (\theta * \mu)^\wedge = \widehat{\theta} \widehat{\mu}.$$

Moreover, $(D\mu)^\wedge = 2\pi i \xi \widehat{\mu}$ and $D\widehat{\mu} = (-2\pi i x \mu)^\wedge$.

Theorem 25.4. $(\sum_{n=-\infty}^{\infty} \delta_n)^\wedge = \sum_{n=-\infty}^{\infty} \delta_n$.

Proof. We can compute that

$$\left\langle \widehat{f}, \left(\sum_{n=-\infty}^{\infty} \delta_n \right)^\wedge \right\rangle = \left\langle f, \sum_{n=-\infty}^{\infty} \delta_n \right\rangle = \sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) = \left\langle \widehat{f}, \sum_{n=-\infty}^{\infty} \delta_n \right\rangle,$$

where we used the Poisson summation formula in the third equality. □