MATH 6422: Algebraic Geometry II

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Contents

Lecture 1

Jan. 13 — Overview and Review

1.1 Course Overview

Remark. This course plans to cover the following topics:

(i) vector bundles and line bundles in algebraic geometry;

(ii) coherent sheaves;

(iii) differentials;

(iv) sheaf cohomology: in particular, we will see that $H^k_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{i+j=k} H^i(X, \wedge^j T^*_X)$;

(v) the Riemann-Roch theorem: if $\omega = f\,dz$ is a rational 1-form on a smooth projective curve C, then

$$(\#\text{ zeroes of }\omega) - (\#\text{ poles of }\omega) = 2\,\mathrm{genus}(C) - 2;$$

(vi) surfaces and toric varieties;

(vii) schemes: for example, $\operatorname{Spec}\mathbb{Z}$ has points corresponding to the primes p and 0.

1.2 Review of Algebraic Geometry I

Remark. Let $k = \overline{k}$ be an algebraically closed field.

Remark (Hilbert's Nullstellensatz)**.** There is a correspondence

$$\begin{aligned} \text{closed subvarieties of } \mathbb{A}^n &\longleftrightarrow \text{radical ideals in } k[x_1, \dots, x_n] \\ Z &\longmapsto I(Z) \\ V(J) &\longleftarrow J. \end{aligned}$$

Under this correspondence, Z being irreducible (resp. a point) corresponds to $I(Z)$ being prime (resp. maximal).

Remark (Zariski topology on $\mathbb{A}^n$)**.** The closed sets in $\mathbb{A}^n$ are of the form $V(J)$, and this induces a Zarisiki topology on any subset of $\mathbb{A}^n$.

Remark (Embedded affine varieties)**.** Let $J \leq k[x_1, \dots, x_n]$. Then we can associate to J a ringed space $(X, \mathcal{O}_X)$ by setting $X := V(J) \subseteq \mathbb{A}^n$ with the Zariski topology, and $\mathcal{O}_X$ to be the sheaf of regular functions on X, i.e. for $U \subseteq X$ open, we have

$$\mathcal{O}_X(U) := \{\varphi : U \to k \mid \varphi \text{ is regular}\}.$$

Here, $\varphi : U \to k$ is *regular* if for each $p \in U$, there exists an open set $p \in U_p \subseteq U$ and $f, g \in k[x_1, \ldots, x_n]$ such that $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$.

Remark (Coordinate ring)**.** The *coordinate ring* of X is

$$A(X) := \mathcal{O}_X(X) \cong k[x_1, \ldots, x_n]/I(X).$$

We also get a version of Hilbert's Nullstellensatz for $A(X)$:

$$\text{closed subsets of } X \longleftrightarrow \text{radical ideals of } A(X).$$

Remark (Distinguished open sets)**.** The *distinguished open sets* of X are

$$D(f) := \{x \in X : f(x) \neq 0\} = X \setminus V(f).$$

These for a basis for X as we vary $f \in A(X)$.

Definition 1.1. An *affine variety* is a ringed space $(X, \mathcal{O}_X)$ (here $\mathcal{O}_X$ is a sheaf of k-valued functions) which is isomorphic to an embedded affine variety.

Example 1.1.1. If $(X, \mathcal{O}_X)$ is an affine variety and $f \in \mathcal{O}_X(X)$, then

$$(D(f), \mathcal{O}_X|_{D(f)})$$

is again an affine variety. To see this, we may assume that $X = V(J) \subseteq \mathbb{A}^n$ with $J \leq k[x_1, \ldots, x_n]$ a radical ideal. Now we can define a map

$$\begin{aligned} D(f) &\longrightarrow V(J, fy - 1) \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^1_y \\ x &\longmapsto (x, 1/f(x)), \end{aligned}$$

which one can check is an isomorphism. Now that this also shows

$$\mathcal{O}_X(D(f)) = A(D(f)) \cong \frac{k[x_1, \ldots, x_n, y]}{(J, fy - 1)} \cong \frac{(k[x_1, \ldots, x_n]/J)[y]}{(\overline{f}y - 1)} \cong A(X)_f.$$

Theorem 1.1. *There is an equivalence of categories*

$$\begin{aligned} \Phi : \text{Aff-var} &\longrightarrow \text{Red-f.g.-}k\text{-alg}^{\text{op}} \\ (X, \mathcal{O}_X) &\longmapsto A(X). \end{aligned}$$

This implies the following:

1. *There is a bijection*

$$\begin{aligned} \mathrm{Hom}_{\text{aff-var}}(X, Y) &\longrightarrow \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^*. \end{aligned}$$

2. *For any reduced finitely generated k-algebra A, there exists an affine variety with $A \cong A(X)$.*

Remark. How can we explicitly define the inverse functor Red-f.g.-k-alg$^{\text{op}} \to$ Aff-var? We can define this as $A \mapsto (X, \mathcal{O}_X)$, where X is the set of maximal ideals of A. Think about what $\mathcal{O}_X$ should be.

Remark (Varieties)**.** A *variety* $(X, \mathcal{O}_X)$ is a ringed space such that

- there exists a finite open cover of X by affine varieties,
- the diagonal Δ_X is closed in $X \times X$.

Example 1.1.2. The following are examples of varieties:

- affine varieties,
- open or closed subsets of varieties,
- $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$.

Remark (Projective spaces)**.** Recall that $\mathbb{P}^n$ has an open cover by

$$U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n.$$

A basis for $\mathbb{P}^n$ by distinguished open sets is given by

$$D(f) = \{[x] \in \mathbb{P}^n : f(x) \neq 0\}$$

with $f \in k[x_0, \ldots, x_n]$ homogeneous.

1.3 Vector and Line Bundles

Definition 1.2. Let X be a variety. A *vector bundle (of rank m)* on X is a variety $\mathbb{E}$ with a morphism $p : \mathbb{E} \to X$ such that

1. $\mathbb{E}_x = p^{-1}(x)$ has the structure of a rank m vector space for every $x \in X$ (i.e. k^m),
2. for every $x \in X$, there exists an open neighborhood $x \in U \subseteq X$ and an isomorphism $p^{-1}(U) \to U \times \mathbb{A}^m$ such that for any $y \in U$, the map $\mathbb{E}_y \to \{y\} \times \mathbb{A}^m$ is an isomorphism of vector spaces, i.e.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "p"'] & & U \times \mathbb{A}^m \arrow[dl, "\text{proj}"] \\ & U & \end{tikzcd}$$

commutes. We will call the map ϕ_U a *trivialization.*

Definition 1.3. A *line bundle* on X is a rank 1 vector bundle.

Remark. A different way to think about this is the following:

1. Given two trivializations $\phi_U : p^{-1}(U) \to U \times \mathbb{A}^m$ and $\phi_V : p^{-1}(V) \to V \times \mathbb{A}^m$, we get a morphism

$$\begin{tikzcd} (U \cap V) \times \mathbb{A}^m \arrow[rr, "\phi_{U,V}"] \arrow[dr, "\phi_V^{-1}"'] & & (U \cap V) \times \mathbb{A}^m \\ & p^{-1}(U \cap V) \arrow[ur, "\phi_U"'] & \end{tikzcd}$$

with $\phi_{U,V} = \phi_U \circ \phi_V^{-1}$. Observe $\phi_{U,V}(x, v) = (x, g_{U,V}(x)v)$ for some $g_{U,V}(x) \in \mathrm{GL}(m, k)$. Furthermore, $g_{U,V} : U \cap V \to \mathrm{GL}(m, k)$ is a morphism. We will call the $g_{U,V}$ *transition functions.*

In the special case where $m = 1$ (so $\mathbb{E}$ is a line bundle and $\mathrm{GL}(1, k) = k^\times$), the map $g_{U,V} : U \cap V \to \mathrm{GL}(m, k)$ is equivalent to the data of a non-vanishing regular function $g_{U,V} : U \cap V \to k$.

2. The data of a vector bundle of rank m is equivalent to the data of
   - an open cover $X = \bigcup_{i \in I} U_i$,
   - and morphisms $g_{i,j} : U_i \cap U_j \to \mathrm{GL}(m, k)$

   such that $g_{i,k} = g_{i,j} g_{j,k}$, $g_{i,j} = g_{j,i}^{-1}$, and $g_{i,i} = \mathrm{id}$.

   To recover the vector bundle, we can glue $\mathbb{E}_i = U_i \times \mathbb{A}^m$ for $i \in I$ via

$$\begin{aligned} \mathbb{E}_{i,j} = (U_i \cap U_j) \times \mathbb{A}^m &\longrightarrow \mathbb{E}_{j,i} = (U_j \cap U_i) \times \mathbb{A}^m \\ (x, v) &\longmapsto (x, g_{i,j}(x) v). \end{aligned}$$

   One can check that this defines a vector bundle $\mathbb{E}$ on X.

Example 1.3.1 (Trivial vector bundle)**.** Define the vector bundle $\mathbb{E} : X \times \mathbb{A}^m \to X$ by $(x, v) \to x$. Given a cover $X = \bigcup_{i \in I} U_i$, we get $g_{i,j} : U_i \cap U_j \to \mathrm{GL}(m, k)$ as $x \mapsto I_m$.

Example 1.3.2 (Trivial line bundle)**.** We will denote the trivial line bundle by $\mathbb{1}_X : X \times \mathbb{A}^1 \to X$.

Lecture 2

Jan. 15 — Vector and Line Bundles

2.1 Vector and Line Bundles, Continued

Example 2.0.1 (Tautological bundle)**.** Let $X = \mathbb{P}^n$ and $\mathbb{L} = \{(\ell, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : x \in \ell\}$. Consider

$$\begin{tikzcd} & \mathbb{L} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^n & & \mathbb{A}^{n+1} \end{tikzcd}$$

The map $q : \mathbb{L} \to \mathbb{A}^{n+1}$ is the blowup. We claim that $p : \mathbb{L} \to \mathbb{P}^n$ is a line bundle. We have:

- $p^{-1}([x]) = \{([x], cx) : c \in k\} \cong kx$, a 1-dimensional vector space;
- let $U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\}$, then we can define

$$\begin{aligned} p^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^1 \\ ([x], y) &\longmapsto ([x], y_i), \end{aligned}$$

  which we claim is a trivialization. To see this, observe that for fixed $[x] \in \mathbb{P}^n$, we have

$$\begin{aligned} \mathbb{L}_{[x]} = \{([x], cx) : c \in k\} &\longrightarrow \{[x]\} \times \mathbb{A}^1 \\ ([x], cx) &\longmapsto ([x], cx_i), \end{aligned}$$

  which is a vector space isomorphism.

We can also compute the transitions functions. Let $U_{i,j} = U_i \cap U_j$. We have

$$\begin{tikzcd} U_{i,j} \times \mathbb{A}^1 \arrow[r, "\phi_j^{-1}"'] \arrow[rr, bend left, "\phi_{i,j}"] & p^{-1}(U_{i,j}) \arrow[r, "\phi_i"'] & U_{i,j} \times \mathbb{A}^1 \\ ([x], t) \arrow[r, mapsto] & ([x], (tx_0/x_j, \ldots, tx_n/x_j)) \arrow[r, mapsto] & ([x], tx_i/x_j). \end{tikzcd}$$

Thus we see that $g_{i,j} = x_i/x_j$. This is called the *tautological bundle*, or $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Example 2.0.2 (Hyperplane bundle, or $\mathcal{O}_{\mathbb{P}^n}(1)$)**.** Consider

$$\begin{aligned} \mathbb{L} := \mathbb{P}^{n+1} \setminus \{[0 : \cdots : 0 : 1]\} &\longrightarrow \mathbb{P}^n \\ [x_0 : \cdots : x_n : x_{n+1}] &\longmapsto [x_0 : \cdots : x_n]. \end{aligned}$$

Then $\mathbb{L}$ is a line bundle with transition functions with respect to $\{U_i\}$ given by $g_{i,j} = x_j/x_i$ (HW).

2.2 Operations on Vector Bundles

Remark. The philosophy is: Every natural operation of vector spaces gives one for vector bundles.

Example 2.0.3 (Direct sum)**.** Let $p : \mathbb{E} \to X$ and $q : \mathbb{F} \to X$ be vector bundles of rank e and f on X, respectively. There exists trivializations with respect to a common open cover $\{U_i\}$ (just take intersections) with transition functions $g_{i,j}$ and $h_{i,j}$ for $\mathbb{E}$ and $\mathbb{F}$, respectively.

Then we define the vector bundle $\mathbb{E} \oplus \mathbb{F} \longrightarrow X$ as follows:

- As a set, it is $r : \mathbb{E} \oplus \mathbb{F} = \{(x, u, v) : (x, u) \in \mathbb{E}, (x, v) \in \mathbb{F}\} \to X$.
- We give $\mathbb{E} \oplus \mathbb{F}$ the structure of a variety by requiring that

$$\begin{aligned} r^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^{e+f} \\ (x, u, v) &\longmapsto (x, \mathrm{pr}_2(\phi_i^E(x, u)), \mathrm{pr}_2(\phi_i^F(x, v))) \end{aligned}$$

  be an isomorphism, where ϕ_i^E and ϕ_i^F are the trivializations of $\mathbb{E}$ and $\mathbb{F}$, and pr_2 is the second projection. This gives a variety structure on $r^{-1}(U_i)$, and one can show that these are consistent on $U_{i,j}$, so that this gives a variety structure on all of $\mathbb{E} \oplus \mathbb{F}$.

Note that the transition functions for $\mathbb{E} \oplus \mathbb{F}$ with respect to $\{U_i\}$ are given by the block matrix

$$\begin{bmatrix} g_{i,j} & 0 \\ 0 & h_{i,j} \end{bmatrix} : U_{i,j} \longrightarrow \mathrm{GL}(e + f, k).$$

Example 2.0.4. Let $\mathbb{E}$ and $\mathbb{F}$ be vector bundles on X of ranks e and f, respectively. Then the following are also vector bundles on X:

1. $\mathrm{Hom}(\mathbb{E}, \mathbb{F})$, of rank ef;
2. $\mathbb{E}^\vee = \mathrm{Hom}(\mathbb{E}, \mathbb{1}_X)$, of rank e;
3. $\mathbb{E} \otimes \mathbb{F}$, of rank ef;
4. $\wedge^k\mathbb{E}$ and $\mathrm{Sym}^d\mathbb{E}$.

Remark. Let $\mathbb{L}, \mathbb{M}$ be line bundles on X with trivializations on $\{U_i\}$ and transition functions $g_{i,j}, h_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$. In this case, we can describe operations on $\mathbb{L}, \mathbb{M}$ more explicitly:

1. $\mathbb{L} \otimes \mathbb{M}$ has transition functions $g_{i,j}h_{i,j}$;
2. $\mathrm{Hom}(\mathbb{L}, \mathbb{M})$ has transition functions $h_{i,j}/g_{i,j}$;
3. $\mathbb{L}^\vee = \mathrm{Hom}(\mathbb{L}, \mathbb{1}_X)$ has transition functions $1/g_{i,j}$;
4. $\mathbb{L}^{\otimes m} = \begin{cases} \mathbb{L}^{\otimes m}, & \text{if } m > 0, \\ \mathbb{1}_X, & \text{if } m = 0, \\ (\mathbb{L}^\vee)^{\otimes -m}, & \text{if } m < 0 \end{cases}$ has transition functions $g_{i,j}^m$.

Example 2.0.5. Define $\mathcal{O}_{\mathbb{P}^n}(m) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}$ with transition functions $(x_j/x_i)^m$ with respect to the standard open cover for $\mathbb{P}^n$.

2.3 Morphisms of Vector Bundles

Remark. Let $p : \mathbb{E} \to X$ and $q : \mathbb{F} \to X$ be vector bundles on X, as before.

Definition 2.1. A *morphism of vector bundles* $\mathbb{E} \to \mathbb{F}$ is a morphism of varieties

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "a"] \arrow[dr, "p"'] & & \mathbb{F} \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

such that the diagram commutes and a is linear on each fiber.

Remark. More concretely, given an open cover $\{U_i\}$ which trivializes both vector bundles, we have

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "a"] \arrow[d, "\phi_i"', "\cong"] & q^{-1}(U_i) \arrow[d, "\psi_j", "\cong"'] \\ U_i \times \mathbb{A}^e \arrow[r] & U_i \times \mathbb{A}^f \end{tikzcd}$$

$$(x, v) \longmapsto (x, a_i(x)v)$$

such that $a_i : U_i \to \mathrm{Hom}(k^e, k^f)$ is regular. On $U_{i,j}$, we have

$$\begin{tikzcd} U_{i,j} \times \mathbb{A}^e \arrow[r, "a_j"] \arrow[d, "g_{i,j}"'] & U_{i,j} \times \mathbb{A}^f \arrow[d, "h_{i,j}"] \\ U_{i,j} \times \mathbb{A}^e \arrow[r, "a_i"'] & U_{i,j} \times \mathbb{A}^f \end{tikzcd}$$

So $h_{i,j}a_j = a_i g_{i,j}$, or equivalently, $a_i = h_{i,j}a_j g_{i,j}^{-1}$.

As a special case when $e = f$, $a : \mathbb{E} \to \mathbb{F}$ is an isomorphism if and only if the a_i are isomorphisms.

Remark. When is a line bundle $\mathbb{L}$ given by the trivialization data $\{U_i, g_{i,j}\}$ isomorphic to $\mathbb{1}_X$? We have

$$\begin{aligned} \mathbb{L} \cong \mathbb{1}_X &\iff \text{if and only if there exists an isomorphism } a : \mathbb{1}_X \to \mathbb{L} \\ &\iff \text{there exist } a_i \in \mathcal{O}_X(U_i)^\times \text{ such that } (a_j/a_i)|_{U_{i,j}} = g_{i,j}. \end{aligned}$$

Definition 2.2. Define the *Picard group* of X to be

$$\mathrm{Pic}\, X := \{\text{line bundles on } X\}/\cong.$$

This is a group with respect to $\otimes$ with $\mathbb{1}_X$ as the identity and $\mathbb{L}^\vee \otimes \mathbb{L} \cong \mathbb{1}_X$.

2.4 Global Sections

Definition 2.3. A *(global) section* of a vector bundle $p : \mathbb{E} \to X$ is a morphism $s : X \to \mathbb{E}$ such that $p \circ s = \mathrm{id}_X$. Note that for $x \in X$, we have $s(x) \in \mathbb{E}_x$.

Example 2.3.1 (Zero section)**.** Let $s : X \to \mathbb{E}$ where $s(x)$ is the zero element in $\mathbb{E}_x$.

Example 2.3.2. Let $\mathbb{E} = \mathbb{1}_X$. Then sections $s : X \to X \times \mathbb{A}^1$ of $\mathbb{E}$ correspond to morphisms $X \to \mathbb{A}^1$, which correspond to regular functions $X \to k$.

Remark (Local description of sections)**.** Let $\{U_i, g_{i,j}\}$ be the trivialization data for $\mathbb{E} \to X$, and let $s : X \to \mathbb{E}$ be a section. On U_i, we have:

$$\begin{tikzcd}
 & U_i \times \mathbb{A}^e \\
U_i \arrow[ur, "x \mapsto (x, s_i(x))"] \arrow[r, "s|_{U_i}"'] & p^{-1}(U_i) \arrow[u, "\phi_i"', "\cong"]
\end{tikzcd}$$

Note that $s_i : U_i \to k^e$ is a regular function (i.e. regular on each coordinate). These maps must satisfy the compatibility condition $s_i = g_{i,j} s_j$, since we have the diagram:

$$\begin{tikzcd}
 & & U_{i,j} \times \mathbb{A}^e \arrow[dd, "(x,v) \mapsto (x, g_{i,j}(v))"] \\
U_{i,j} \arrow[urr, bend left, "(x, s_j(x))"] \arrow[drr, bend right, "(x, s_i(x))"'] \arrow[r, "s|_{U_{i,j}}"] & p^{-1}(U_{i,j}) \arrow[ur, "\phi_j"] \arrow[dr, "\phi_i"'] & \\
 & & U_{i,j} \times \mathbb{A}^e
\end{tikzcd}$$

Example 2.3.3. We can use the above compatibility condition to compute the global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Write $\mathbb{P}^1_{x_0:x_1} = U_0 \cup U_1$. Given a section $s : \mathbb{P}^1 \to \mathcal{O}(1)$, we get regular functions

$$s_0 : U_0 \longrightarrow k$$
$$s_1 : U_1 \longrightarrow k$$

satisfying $(x_1/x_0)s_1 = s_0$ $(*)$. We can write

$$s_0 = \sum_{m \geq 0} a_m (x_1/x_0)^m \quad \text{and} \quad s_1 = \sum_{m \geq 0} b_m (x_0/x_1)^m$$

with $a_m, b_m \in k$ (finitely many nonzero). Then $(*)$ implies that

$$a_0 + a_1(x_1/x_0) + \cdots = (x_1/x_0)(b_0 + b_1(x_0/x_1) + \cdots),$$

so $a_0 = b_1$, $a_1 = b_0$, and all other terms are 0. So we can relate s to a linear form

$$f = a_0 x_0 + a_1 x_1,$$

where $s_0 = (1/x_0)f$ and $s_1 = (1/x_1)f$.

Lecture 3

Jan. 20 — Sections

3.1 Global Sections, Continued

Definition 3.1. Let $\Gamma(X, \mathbb{E}) := \{\text{sections of } \mathbb{E} \to X\}$, which has the structure of a k-vector space by

$$(s+t)(x) = s(x) + t(x) \quad \text{and} \quad (cs)(x) = cs(x)$$

for $s, t \in \Gamma(X, \mathbb{E})$ and $c \in k$.

Example 3.1.1. One can check that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong k[x_0, \ldots, x_n]_d$ (HW). For example, for $d < 0$, we have $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) = \{0\}$, and for $d = 0$, we have

$$\Gamma(\mathbb{P}^n, \mathcal{O}(0)) = \Gamma(\mathbb{P}^n, \mathbb{1}_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k.$$

For $d = 1$, we can define an isomorphism

$$\begin{aligned} k[x_0, \ldots, x_n]_1 &\xrightarrow{\cong} \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \\ f &\longmapsto \text{section } s : \mathbb{P}^n \to \mathcal{O}(1) \text{ given by } s_i = f/x_i. \end{aligned}$$

Note that $(x_j/x_i)s_j = s_i$ holds, so this is a section. An alternative perspective is that s corresponds to

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^{n+1} \setminus \{[0 : \cdots : 0 : 1]\} \\ x &\longmapsto [x_0 : \cdots : x_n : f(x)]. \end{aligned}$$

3.2 Morphisms and Sections

Definition 3.2. Given a section $s : X \to \mathbb{E}$, its *vanishing locus* is

$$V(s) := \{s = 0\} = \{x \in X : s(x) = 0\}.$$

Using a trivializing cover, one can check that $V(s)$ is closed in X.

Example 3.2.1. For a section $s : \mathbb{P}^n \to \mathcal{O}(1)$ corresponding to $f \in k[x_0, \ldots, x_n]_1$, we have $V(s) = V_{\mathbb{P}^n}(f)$.

Remark. Recall that there is a bijection

$$\begin{aligned} \{\text{morphisms } X \to \mathbb{A}^n\} &\longleftrightarrow \{f_1, \ldots, f_n \in \mathcal{O}_X(X)\} \\ [f : X \to \mathbb{A}^n] &\longmapsto [f_1 = f^*x_1, \ldots, f_n = f^*x_i \in \mathcal{O}_X(X)] \\ [x \mapsto (f_1(x), \ldots, f_n(x))] &\longleftarrow [f_1, \ldots, f_n \in \mathcal{O}_X(X)]. \end{aligned}$$

We want a similar statement for $\mathbb{P}^n$.

Definition 3.3. Given a line bundle $\mathbb{L} \to X$ and $s_0, \ldots, s_n \in \Gamma(X, \mathbb{L})$, they are *nowhere vanishing* if

$$V(s_0) \cap \cdots \cap V(s_n) = \varnothing.$$

Example 3.3.1. For $\mathcal{O}(1)$, the sections $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ are nowhere vanishing.

Remark. If $s_0, \ldots, s_n \in \Gamma(X, \mathbb{L})$ are nowhere vanishing, then we get a morphism

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \cdots : s_n(x)]. \end{aligned}$$

Note that $(s_0(x), \ldots, s_n(x))$ is a well-defined point in $\mathbb{A}^{n+1}$ up to scaling. One can check that this map is a morphism by working locally.

Example 3.3.2 (Linear maps)**.** Let $X = \mathbb{P}^n$ and $\mathbb{L} = \mathcal{O}(1)$.

(i) $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ gives $\mathrm{id} : \mathbb{P}^n \to \mathbb{P}^n$.

(ii) For $A \in \mathrm{GL}_{n+1}(k)$, we get a map

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

given by $Ax_0, \ldots, Ax_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Remark. Now given a morphism $X \to \mathbb{P}^n$, we want to get a line bundle with sections.

Definition 3.4 (Pullback)**.** Let $p : \mathbb{E} \to X$ be a vector bundle and $f : Y \to X$ a morphism. Define

$$f^*\mathbb{E} = \{(e, y) : e \in \mathbb{E}, y \in Y \text{ with } p(e) = f(y)\} \longrightarrow Y.$$

One can show that this has the structure of a vector bundle in a natural way.

Remark. An alternative way to define the pullback is to choose trivialization data $(U_i, g_{i,j})$ for $\mathbb{E} \to X$. Then we can define $f^* : \mathbb{E} \to Y$ to be the vector bundle with trivialization data $(f^{-1}(U_i), f^*g_{i,j})$.

Remark. Now to go in reverse, given a morphism $X \to \mathbb{P}^n$ and nowhere vanishing sections $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, we get nowhere vanishing sections

$$f^*x_0, \ldots, f^*x_n \in \Gamma(X, f^*\mathcal{O}(1)).$$

We can define the pullback of a section in one of two ways: by $f^*(x_i)(a) = (x_i(f(a)), a) \in f^*\mathcal{O}(1))$ for $a \in X$ or by using trivializing covers.

Remark. Using the above, we get a bijection

$$\{\text{morphisms } X \to \mathbb{P}^n\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \to X \text{ with } s_0, \ldots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\}.$$

Note that we should consider the right-hand side up to isomorphism of the line bundle. When do $\mathbb{L} \to X$ and $s_0, \ldots, s_n \in \Gamma(X, \mathbb{L})$ give an injective morphism (or an embedding)?

Definition 3.5. Given a vector bundle $\mathbb{E} \to X$, we get a sheaf of abelian groups $\mathcal{E}$ on X by

$$\mathcal{E}(U) := \{\text{sections of } p^{-1}(U) \to U\}$$

for $U \subseteq X$ open. For $V \subseteq U \subseteq X$ open, the restriction map is given by

$$\begin{aligned} \mathcal{E}(U) &\longrightarrow \mathcal{E}(V) \\ s &\longmapsto s|_V. \end{aligned}$$

We call $\mathcal{E}$ the *sheaf of sections* of $\mathbb{E}$. Also note that $\mathcal{E}(U)$ has the structure of an $\mathcal{O}_X(U)$-module. We will see that this gives rise to the structure of an $\mathcal{O}_X$-module.

3.3 Review of Sheaves

Definition 3.6. A *presheaf* of abelian groups $\mathcal{F}$ on a topological space X is the data of:

- for $U \subseteq X$ open, an abelian group $\mathcal{F}(U)$ (with $\mathcal{F}(\varnothing) = 0$),
- for $V \subseteq U \subseteq X$ open, a group homomorphism $p_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$.

Remark. Note the following:

1. We may replace abelian groups in the above definition by rings, sets, R-modules, etc.
2. We denote $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$, whose elements are called *sections*.
3. $s|_V := p_{V,U}(s)$ is called the *restriction* for $s \in \mathcal{F}(U)$ and $V \subseteq U \subseteq X$ open.
4. We may view $\mathcal{F}$ as a functor $\mathrm{Open}_X \to$ Ab-gps given by $U \mapsto \mathcal{F}(U)$.

Definition 3.7. For $\mathcal{F}$ a presheaf on X and $x \in X$, the *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \varinjlim_{U \ni x \text{ open}} \mathcal{F}(U) = \{(s, U) : s \in \mathcal{F}(U)\}/\sim.$$

Example 3.7.1. The following are examples of presheaves:

1. Let M be a smooth manifold. Then
   - $\mathcal{O}_M$ = sheaf of smooth $\mathbb{R}$-valued functions on M,
   - $\mathcal{E}$ = sheaf of sections of a vector bundle $\mathbb{E} \to M$.
2. Let X be an algebraic variety, $\mathbb{E} \to X$ a vector bundle, and $Z \subseteq X$ closed. Then
   - $\mathcal{O}_X$ and $\mathcal{E}$ are sheaves,
   - $\mathcal{I}_Z$ = ideal sheaf of Z, given by $\mathcal{I}_Z(U) = \{\varphi \in \mathcal{O}_X(U) : \varphi|_Z = 0\}$.
3. Let X be a topological space and A an abelian group.
   - $\underline{A}^{\text{pre}}$ given by $U \mapsto \{$constant functions $U \to A\}$, i.e. $\underline{A}^{\text{pre}}(U) \cong A$ for $U \neq \varnothing$,
   - $\underline{A}$ given by $U \mapsto \{$locally constant functions $U \to A\}$,
   - $i_p A$ = skyscraper sheaf, given by $U \mapsto \begin{cases} A & \text{if } p \in U, \\ 0 & \text{otherwise.} \end{cases}$

Definition 3.8. A presheaf $\mathcal{F}$ is a *sheaf* if for any

- open set $U \subseteq X$,
- open cover $U = \bigcup_{i \in I} U_i$,
- and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_{i,j}} = s_j|_{U_{i,j}}$ for all $i, j \in I$,

then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for every $i \in I$.

Remark. The presheaf $\underline{A}^{\text{pre}}$ is not a sheaf in general. All other examples above are sheaves.

Definition 3.9. A *morphism* of (pre)sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on a topological space X is the data of group homomorphisms $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each $U \subseteq X$ open such that for all $V \subseteq U \subseteq X$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\text{res}"'] & \mathcal{G}(U) \arrow[d, "\text{res}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

Example 3.9.1. Let X be a variety.

1. If $a : \mathbb{E} \to \mathbb{F}$ is a morphism of vector bundles on X, then we get a morphism of sheaves $\mathcal{E} \to \mathcal{F}$ by $s \mapsto a \circ s \in \mathcal{F}(U)$ for $s \in \mathcal{E}(U)$.

2. A closed subvariety $Z \subseteq X$ induces a morphism $\mathcal{I}_Z \to \mathcal{O}_X$ given by inclusion.

Remark. Given a morphism of (pre)sheaves and $\varphi : \mathcal{F} \to \mathcal{G}$ and $p \in X$, we get an induced morphism

$$\begin{aligned} \mathcal{F}_p &\longrightarrow \mathcal{G}_p \\ (s, U) &\longmapsto (\varphi(s), U). \end{aligned}$$

Lecture 4

Jan. 22 — Sheaves

4.1 Sheafification

Theorem 4.1 (Sheafification)**.** *For a presheaf $\mathcal{F}$ on a topological space X, there exists a morphism to a sheaf $i : \mathcal{F} \to \mathcal{F}^+$ such that for any morphism to a sheaf $g : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $g^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $g = g^+ \circ i$, i.e. the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "g"] \arrow[d, "i"'] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, dashed, "g^+"'] & \end{tikzcd}$$

In the above, $\mathcal{F}^+$ is called the sheafification *of $\mathcal{F}$, and the pair $(i, \mathcal{F}^+)$ is unique up to isomorphism (as a consequence of the universal property).*

Proof. We first define $\mathcal{F}^+(U) = \{t : U \to \bigsqcup_{p \in X} \mathcal{F}_p : (1) \text{ and } (2) \text{ hold}\}$, where

1. $t(p) \in \mathcal{F}_p$;
2. for any $x \in X$, there is an open set $x \in V_x \subseteq U$ with $s \in \mathcal{F}(V_x)$ such that $t(p) = s_p$ for all $p \in V_x$.

It is straightforward to see that $\mathcal{F}^+$ is a sheaf and that

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathcal{F}^+(U) \\ s &\longmapsto (X \ni p \mapsto s_p \in \mathcal{F}_p). \end{aligned}$$

gives a morphism $i : \mathcal{F} \to \mathcal{F}^+$. Now we check the universal property. Given a morphism $g : \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a sheaf, we need to define $g^+ : \mathcal{F}^+ \to \mathcal{G}$. Fix $t \in \mathcal{F}^+(U)$. By definition, there exists an open cover $\{U_i\}$ of U and $s_i \in \mathcal{F}(U_i)$ such that $t(p) = (s_i)_p \in \mathcal{F}_p$ for all $p \in U_i$. Set $t_i' := g(t_i)$. Note that

$$(t_i')_p = g_p(t_p) = (t_j')_p \in \mathcal{G}_p$$

for every $p \in U_i \cap U_j$. Since $\mathcal{G}$ is a sheaf, we get $t_i'|_{U_i \cap U_j} = t_j'|_{U_i \cap U_j}$. Thus there exists a unique $t' \in \mathcal{G}(U)$ such that $t'|_{U_i} = t_i$ for every $i \in I$. Then we can set $g^+(t) = t'$. One can check as an exercise that this gives a morphism $\mathcal{F}^+ \to \mathcal{G}$ satisfying the universal property. $\square$

Example 4.0.1. We have the following:

1. If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.
2. For an abelian group A and topological space X, we have $(\underline{A}^{\text{pre}})^+ \cong \underline{A}$.

Remark. We have the following:

1. If $p \in X$, the induced morphism on stalks $i_p : \mathcal{F}_p \to \mathcal{F}_p^+$ is an isomorphism for all $p \in X$.

   To construct the inverse map, consider $\mathcal{F}_p^+ \to \mathcal{F}_p$ defined by $(t, U) \mapsto t_p$ for $t \in \mathcal{F}^+(U)$. Check as an exercise that this is well-defined and is inverse to i_p.

2. If $\mathcal{F} \subseteq \mathcal{G}$ is a subpresheaf (i.e. $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and $\rho^{\mathcal{F}}_{V,U} = \rho^{\mathcal{G}}_{V,U}|_{\mathcal{F}(U)}$ for all $V \subseteq U \subseteq X$) and $\mathcal{G}$ is a sheaf, then we could alternatively define $\mathcal{F}^+$ as

$$\mathcal{F}^+(U) = \{s \in \mathcal{G}(U) : \text{for all } x \in X, \text{ there exists } x \in U_x \subseteq U \text{ open such that } s|_{U_x} \in \mathcal{F}(U_x)\}.$$

4.2 Kernel, Image, Cokernel for Sheaves

Remark. We want the following notions for sheaves:

- kernel, image, cokernel;
- short exact sequences;
- injectivity and surjectivity.

Example 4.0.2. We want the following to be short exact sequences of sheaves:

- for X a variety and $Z \hookrightarrow X$ a closed subvariety,

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

- for M a complex manifold (e.g. $\mathbb{C}^n$) with $\mathcal{O}_M$ the sheaf of $\mathbb{C}$-valued holomorphic functions,

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i \times} \mathcal{O}_M \xrightarrow{\varphi \mapsto e^{\varphi}} \mathcal{O}_M^\times \longrightarrow 0$$

Remark. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X.

Definition 4.1. The *kernel* of φ is $(\ker \varphi)(U) = \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$.

Remark (Properties of the kernel)**.** It is straightforward to check that $\ker \varphi$ is a sheaf. Moreover:

1. $\ker \varphi$ satisfies the following universal property: For any morphism to a sheaf α such that $\varphi \circ \alpha = 0$, there exists a unique morphism α' such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "\alpha"] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\alpha'"'] & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \\ & \ker \varphi \arrow[u] & \end{tikzcd}$$

   To see this, use the universal property of the kernel in the category of abelian groups.

2. Since filtered limits are exact, we have $(\ker \varphi_p) = (\ker \varphi)_p$ for all $p \in X$.

Lemma 4.1 (Injectivity for sheaves)**.** *The following are equivalent:*

1. *$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all $U \subseteq X$ open;*
2. *$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$.*

We say that φ is injective *if either of these equivalent conditions hold.*

Proof. ($1 \Rightarrow 2$) This is clear.

($2 \Rightarrow 1$) Fix $s \in \mathcal{F}(U)$ with $\varphi(U)(s) = 0$. Then

$$\varphi_p(s_p) = (\varphi(U)(s))_p = 0$$

for all $p \in X$, so $s_p = 0$ for all $p \in U$, so $s = 0$ by homework from Algebraic Geometry I. □

Example 4.1.1 (Subtleties for the image)**.** Consider the following:

1. Let $\varphi : \mathcal{O}_{\mathbb{C}^n} \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}^n}^\times$. Then $U \mapsto \operatorname{im}(\mathcal{O}_{\mathbb{C}^n}(U) \to \mathcal{O}_{\mathbb{C}^n}^\times(U))$ is a presheaf but not a sheaf. This is because logarithms only exist locally.
2. Define $\varphi : \mathcal{O}_{\mathbb{P}^n} \to i_{p_1}\underline{k} \oplus i_{p_2}\underline{k}$ by $f \mapsto (f(p_1), f(p_2))$. Again $U \mapsto \operatorname{im}(\varphi(U))$ is not a sheaf.

Definition 4.2. Let $\widetilde{\operatorname{im}}\,\varphi = (U \mapsto \operatorname{im}(\varphi(U))$. This is a presheaf and $\widetilde{\operatorname{im}}\,\varphi \subseteq \mathcal{G}$. Then the *image* of φ is $\operatorname{im}\varphi = (\widetilde{\operatorname{im}}\,\varphi)^+$. By Remark 4.1, we can equivalently define

$$(\operatorname{im}\varphi)(U) = \{s \in \mathcal{G}(U) : \text{there exists cover } \{U_i\} \text{ of } U \text{ such that } s|_{U_i} \in \operatorname{im}(\mathcal{F}(U_i) \to \mathcal{G}(U_i))\}.$$

Remark. We have $\operatorname{im}(\varphi_x) \cong (\widetilde{\operatorname{im}}\,\varphi)_x \cong (\operatorname{im}\varphi)_x$, where the first isomorphism is because filtered direct limits are exact and the second isomorphism is because sheafification preserves stalks.

Definition 4.3. Let $\widetilde{\operatorname{coker}}(\varphi) = (U \mapsto \operatorname{coker}(\varphi(U)))$. Then the *cokernel* of φ is $\operatorname{coker}(\varphi) = (\widetilde{\operatorname{coker}}(\varphi))^+$.

Remark. We have the following:

1. $\operatorname{coker}(\varphi)_x \cong \operatorname{coker}(\varphi_x)$ (similar to above).
2. $\operatorname{coker}(\varphi)$ satisfies the universal property of the cokernel:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[rr, bend left, "0"] & \mathcal{G} \arrow[r, "p_0"] \arrow[d] & \mathcal{G}' \\ & \widetilde{\operatorname{coker}}(\varphi) \arrow[r] \arrow[ur, dashed] & \operatorname{coker}(\varphi) \arrow[u, dashed] \end{tikzcd}$$

3. For a subsheaf $\mathcal{F}' \subseteq \mathcal{F}$, we can define the *quotient sheaf* $\mathcal{F}/\mathcal{F}' = \operatorname{coker}(\mathcal{F}' \hookrightarrow \mathcal{F})$.
4. By the universal property of the cokernel, we get natural maps

$$\begin{tikzcd} \ker\varphi \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \operatorname{im}\mathcal{F} \\ & \mathcal{F}/\ker\varphi \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

As the following diagram commutes,

$$\begin{tikzcd}
(\mathcal{F}/\ker\varphi)_p \arrow[r, "\alpha_p"] \arrow[d, "\cong"] & (\operatorname{im}\varphi)_p \arrow[d, "\cong"'] \\
\mathcal{F}_p/(\ker\varphi)_p \arrow[r, "\cong"'] & \operatorname{im}(\varphi_p)
\end{tikzcd}$$

α_p is an isomorphism for all $p \in X$. So by HW, α is an isomorphism. So $\mathcal{F}/\ker\varphi \cong \operatorname{im}\varphi$.

Lemma 4.2 (Surjectivity for sheaves)**.** *The following are equivalent:*

1. $\operatorname{coker}\varphi = 0$*;*
2. $\operatorname{im}\varphi = \mathcal{G}$*;*
3. $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ *is surjective for all* $x \in X$*.*

We say that φ *is* surjective *if any of these equivalent conditions hold.*

Proof. ($3 \Leftrightarrow 1$) We have (3) if and only if $\operatorname{coker}(\varphi_x) = 0$ for all $x \in X$, if and only if $(\operatorname{coker}\varphi)_x = 0$ for all $x \in X$, if and only if (1).

($3 \Leftrightarrow 2$) We have (3) if and only if $\operatorname{coker}(\varphi_x) = 0$ for all $x \in X$, if and only if $\operatorname{im}\varphi_x = \mathcal{G}_x$ for all $x \in X$, if and only if $(\operatorname{im}\varphi)_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$, if and only if (2) by HW. $\square$

Remark. Note that if $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all $U \subseteq X$, then φ is surjective. However, the converse is false in general.

Definition 4.4. A sequence of morphisms of sheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is *exact* at $\mathcal{G}$ if $\ker g = \operatorname{im} f$.

Lemma 4.3. *The following are equivalent:*

1. $\ker g = \operatorname{im} f$*;*
2. $\ker g_x = \operatorname{im} f_x$ *for all* $x \in X$*.*

Proof. Similar to above. $\square$

4.3 Constructions with Sheaves

Definition 4.5. Let $\mathcal{F}_1, \mathcal{F}_2$ be sheaves on X. Then $\mathcal{F}_1 \oplus \mathcal{F}_2$ is a sheaf defined by

$$U \longmapsto \mathcal{F}_1(U) \oplus \mathcal{F}_2(U).$$

This is a *biproduct* in the category of sheaves.

Example 4.5.1. Let X be a variety with connected components $X_1, \ldots, X_n$. Then

$$\mathcal{O}_X \cong \mathcal{O}_{X_1} \oplus \cdots \oplus \mathcal{O}_{X_n}$$

as sheaves of abelian groups.

Definition 4.6. Let $U \subseteq X$ be open. Then $\mathcal{F}_i|_U$ is a sheaf on U given by $V \mapsto \mathcal{F}_i(V)$

Definition 4.7. $\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$ is the sheaf $U \mapsto \mathrm{Hom}^{\text{sheaves}}(\mathcal{F}_1|_U, \mathcal{F}_2|_U)$.

Definition 4.8 (Gluing)**.** Let $\{U_i\}$ be an open cover of X with a sheaf $\mathcal{F}_i$ on each U_i and isomorphisms $\alpha_{i,j} : \mathcal{F}_j|_{U_{i,j}} \to \mathcal{F}_i|_{U_{i,j}}$ such that $\alpha_{i,j} = \alpha_{j,i}^{-1}$, $\alpha_{i,j} \circ \alpha_{j,k} = \alpha_{i,k}$, and $\alpha_{i,i} = \mathrm{id}$. Then there exists a sheaf $\mathcal{F}$ with isomorphisms $\beta_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}|_{U_{i,j}} \arrow[r, "\mathrm{id}"] \arrow[d, "\beta_j"'] & \mathcal{F}|_{U_{i,j}} \arrow[d, "\beta_i"] \\
\mathcal{F}_i|_{U_{i,j}} \arrow[r, "\alpha_{i,j}"'] & \mathcal{F}_j|_{U_{i,j}}
\end{tikzcd}$$

One can define $\mathcal{F}$ as follows and check that it satisfies the above properties:

$$\mathcal{F}(U) = \{(s_i)_{i \in I} : s_i \in \mathcal{F}(U_i) \text{ and } \alpha_{i,j}(s_j|_{U_{i,j}}) = s_i\}.$$

Lecture 5

Jan. 27 — $\mathscr{O}_X$-Modules

5.1 Sheaves and Continuous Maps

Remark. The category Sh_X of sheaves of abelian groups on a topological space X is an abelian category, i.e. it has or satisfies the following:

- zero object (the $\underline{0}$ sheaf);
- $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is an abelian group and composition is bilinear;
- finite biproducts exist;
- kernels and cokernels exist;
- the image coincides with the coimage.

Remark. For the rest of this section, let $f : X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y.

Definition 5.1. The *pushforward* $f_*\mathcal{F}$ is the sheaf on Y defined by

$$V \longmapsto \mathcal{F}(f^{-1}(V)).$$

Example 5.1.1. We have the following:

- If $i : \{p\} \hookrightarrow X$ and A is an abelian group, then $i_*\underline{A}$ is the skyscraper sheaf on X at p.
- If X is a variety and $i : Z \hookrightarrow X$ with Z a closed subvariety, then

$$i_*\mathcal{O}_Z(U) = \{\varphi : U \cap Z \to k : \varphi \text{ is regular}\}.$$

Definition 5.2. Let $\widetilde{f^{-1}\mathcal{G}}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, which is a presheaf. The *pullback* is $f^{-1}\mathcal{G} := (\widetilde{f^{-1}\mathcal{G}})^+$.

Example 5.2.1. Let $f : \{p\} \hookrightarrow X$. Then $\widetilde{f^{-1}\mathcal{G}} \cong \underline{\mathcal{G}_p}$, which is a sheaf, so $f^{-1}\mathcal{G} \cong \widetilde{f^{-1}\mathcal{G}} \cong \underline{\mathcal{G}_p}$.

Example 5.2.2. If $i : U \hookrightarrow X$ is the inclusion of an open set, then $i^{-1}\mathcal{F} \cong \mathcal{F}|_U$.

Remark. The pushforward f_* and pullback f^{-1} are functors:

$$\begin{tikzcd} \mathrm{Sh}_X \arrow[r, bend left, "f_*"] & \mathrm{Sh}_Y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

How are f_* and f^{-1} related? There are natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ induced by:

1. For $U \subseteq X$, define

$$\widetilde{f^{-1}} f_*\mathcal{F}(U) = \varinjlim_{V \supseteq f(U) \text{ open}} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U)$$
$$s \longmapsto s|_U.$$

2. For $V \subseteq Y$, define

$$\mathcal{G}(V) \longrightarrow \varinjlim_{V \supseteq V' \supseteq f(f^{-1}(V)) \text{ open}} \mathcal{G}(V') = (f_*\widetilde{f^{-1}}\mathcal{G})(V).$$

Note that $V \supseteq f(f^{-1}(V))$, so we can add the $V \supseteq V' \supseteq f(f^{-1}(V))$ condition.

Another way to think about this is via adjoints.

Proposition 5.1. *For $\mathcal{F} \in \mathrm{Sh}_X$ and $\mathcal{G} \in \mathrm{Sh}_Y$, there exist functorial bijections*

$$\mathrm{Hom}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathrm{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}),$$

i.e. (f^{-1}, f_*) is an adjoint pair.

Proof. Given $\phi : f^{-1}\mathcal{G} \to \mathcal{F}$, using map (2) from above we get

$$\mathcal{G} \xrightarrow{(2)} f_*f^{-1}\mathcal{G} \xrightarrow{f_*\phi} f_*\mathcal{F}.$$

Similarly, given $\psi : \mathcal{G} \to f_*\mathcal{F}$, using map (1) from above we get

$$f^{-1}\mathcal{G} \xrightarrow{f^{-1}\psi} f^{-1}f_*\mathcal{F} \xrightarrow{(1)} \mathcal{F}.$$

One can (tediously) check that this gives a bijection. □

Remark. The following are consequences of adjointness:

- f_* is left exact, i.e. given an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

  in Sh_X, we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F}' \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{F}''.$$

- f^{-1} is right exact (defined similarly with the left 0 missing).

One can also directly check these properties from the definitions.

5.2 Sheaves of $\mathcal{O}_X$-Modules

Remark. To use tools from commutative algebra, we want to consider modules. For the rest of this section, let X be a topological space with a sheaf of rings $\mathcal{O}_X$ (e.g. X a variety with $\mathcal{O}_X$ the sheaf of regular functions, or M a complex manifold with $\mathcal{O}_M$ the sheaf of holomorphic functions).

Definition 5.3. A *(pre)sheaf of $\mathcal{O}_X$-modules* is a (pre)sheaf $\mathcal{F}$ on X such that for each $U \subseteq X$, $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$-module compatible with restriction, i.e. such that

$$(a \cdot s)|_V = a|_V \cdot s|_V$$

for $V \subseteq U \subseteq X$ open, $a \in \mathcal{O}_X(U)$, and $s \in \mathcal{F}(U)$.

Remark. We often say just "$\mathcal{O}_X$-module" to mean a "sheaf of $\mathcal{O}_X$-modules."

Definition 5.4. A *morphism* of (pre)sheaves of $\mathcal{O}_X$-modules $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism of presheaves such that $\mathcal{M}(U) \to \mathcal{N}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules for all $U \subseteq X$.

Example 5.4.1. We have the following:

1. $\mathcal{O}_X$ has the structure of an $\mathcal{O}_X$-module (similar to how a ring A has the structure of an A-module).

2. Let X be a variety and $p : \mathbb{E} \to X$ a vector bundle. Let $\mathcal{E}$ be the sheaf of sections of p, with

$$[f \cdot s : U \to p^{-1}(U)] \in \mathcal{E}(U)$$

as the product of $f \in \mathcal{O}_X(U)$ and $[s : U \to p^{-1}(U)] \in \mathcal{E}(U)$. Then $\mathcal{E}$ is an $\mathcal{O}_X$-module.

2'. Let $\mathbb{E} = \mathbb{1}_X$, then $\mathcal{E}(U) \cong \mathcal{O}_X(U)$ as $\mathcal{O}_X(U)$-modules. As the isomorphism is compatible with restriction, we have $\mathcal{E} \cong \mathcal{O}_X$.

3. Let $\mathcal{F}_1, \mathcal{F}_2$, be (pre)sheaves of $\mathcal{O}_X$-modules. Then so is $\mathcal{F}_1 \oplus \mathcal{F}_2$.

2". If $\mathbb{E}$ is a trivial vector bundle of rank e, then $\mathcal{E} \cong \mathcal{O}_X^{\oplus e}$.

4. If $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}^+$ is naturally a sheaf of $\mathcal{O}_X$-modules (use the definition of $\mathcal{F}^+$ in the proof).

5. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of $\mathcal{O}_X$-modules, then $\ker\varphi, \operatorname{im}\varphi, \operatorname{coker}\varphi$ are sheaves of $\mathcal{O}_X$-modules (use that $\ker\varphi, \widetilde{\operatorname{im}}\,\varphi, \widetilde{\operatorname{coker}}\,\varphi$ are presheaves of $\mathcal{O}_X$-modules and then use (4)).

   Furthermore, the category $\mathrm{Mod}_{\mathcal{O}_X}$ of sheaves of $\mathcal{O}_X$-modules is an abelian.

6. We can define the usual constructions on $\mathcal{O}_X$-modules: $\otimes$, Sym^d, $\wedge^d$, etc.

Example 5.4.2. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Then their tensor product $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Definition 5.5. An $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* of rank e if for any $p \in X$, there exists $p \in U \subseteq X$ open such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus e}$. If $e = 1$, then we say that $\mathcal{F}$ is *invertible*.

Example 5.5.1. Let X be a variety and $p : \mathbb{E} \to X$ a vector bundle of rank e. For $p \in X$, there exists $p \in U \subseteq X$ open such that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[d, "p"'] & U \times \mathbb{A}^e \arrow[dl, "\mathrm{pr}_1"] \\ U & \end{tikzcd}$$

Then $\mathcal{E}|_U \cong$ sheaf of sections of $U \times \mathbb{A}^e \cong \mathcal{O}_U^{\oplus e}$.

Remark (Transition functions)**.** Let $e = 1$ for simplicity, and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank e. Then there exists an open cover $\{U_i\}$ of X with isomorphisms $\alpha_i : \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus e}$. So on $U_{i,j} = U_i \cap U_j$, we get isomorphisms $\alpha_{i,j} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{i,j}}^{\oplus e} \to \mathcal{O}_{U_{i,j}}^{\oplus e}$.

Lecture 6

Jan. 29 — $\mathcal{O}_X$-Modules, Part 2

6.1 More on $\mathcal{O}_X$-Modules

Remark. Recall that we have operations $\otimes$, $\oplus$, Sym^d, $\wedge^d$, $\mathcal{H}om(\cdot,\cdot)$ on $\mathcal{O}_X$-modules.

Example 6.0.1. The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is the sheafification of

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U).$$

This is again an $\mathcal{O}_X$-module.

Example 6.0.2. We have $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X)$.

Exercise 6.1. The invertible sheaves on X up to isomorphism forms a group with multiplication given by $\otimes$, identity $\mathcal{O}_X$, and inverse $\mathcal{L}^{-1} = \mathcal{L}^\vee$.

Remark (Transition data)**.** Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then there exists an open cover $\{U_i\}$ of X and isomorphisms $\alpha_i : \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$, so we get isomorphisms

$$\alpha_{i,j} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{i,j}} \to \mathcal{O}_{U_{i,j}}.$$

For this to be an isomorphism, we must have $\alpha_{i,j}(U_{i,j})(1) = g_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$.

Proposition 6.1. *If X is a variety, then there is a bijection*

$$\begin{aligned}\{\textit{line bundles on } X\}/\cong &\longrightarrow \{\textit{invertible sheaves on } X\}/\cong \\ \mathbb{L} &\longmapsto \mathcal{L}\end{aligned}$$

Proof. To get the reverse map, fix an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with trivialization data $(U_i, g_{i,j})$. Send it to the line bundle with the same trivialization data. Check that this is well-defined as an exercise.

To show that this gives an inverse, it suffices to show that if $\mathbb{L}$ is a line bundle with trivialization data $(U_i, g_{i,j})$, then the sheaf $\mathcal{L}$ of sections of $\mathbb{L}$ has the same trivialization data. The trivializations

$$\mathbb{L}|_{U_i} \xrightarrow[\cong]{\phi_i} U_i \times \mathbb{A}^1$$

give isomorphisms $U_{i,j} \times \mathbb{A}^1 \to U_{i,j} \times \mathbb{A}^1$ by $(x,v) \mapsto (x, g_{i,j}(x)v)$. We get an isomorphism

$$\alpha_i : \mathcal{L}|_{U_i} \xrightarrow[\cong]{\phi_i} \mathcal{O}_{U_i}$$

where $e_i := \alpha_i^{-1}(1) = [U_i \xrightarrow{x \mapsto (x,1)} U_i \times \mathbb{A}^1 \xrightarrow{\phi_i^{-1}} \mathbb{L}|_{U_i}]$. Now we have

$$\begin{aligned} \mathcal{O}_{U_{i,j}} &\xrightarrow{\alpha_j^{-1}} \mathcal{L}|_{U_{i,j}} \xrightarrow{\alpha_i} \mathcal{O}_{U_{i,j}} \\ 1 &\longmapsto e_j = e_i g_{i,j} \longmapsto g_{i,j}. \end{aligned}$$

So we get the same transition functions $(U_i, g_{i,j})$ for $\mathcal{L}$. □

Remark. Given a morphism of rings $\phi : A \to B$, we have functors

$$\begin{tikzcd} \mathrm{Mod}_B \arrow[r, bend left, "\Phi"] & \mathrm{Mod}_A \arrow[l, bend left, "\Psi"] \end{tikzcd}$$

given as follows:

1. *Extension of scalars*: $\mathrm{Mod}_A \ni M \longmapsto M \otimes_A B \in \mathrm{Mod}_B$, where the multiplication by B is
$$c(m \otimes b) = m \otimes (cb).$$
For example, if $M = A^{\oplus I}$, then $M \otimes_A B = B^{\oplus I}$.
2. *Restriction of scalars*: $\mathrm{Mod}_B \ni N \longmapsto N_A \in \mathrm{Mod}_A$, where $N_A := N$ as abelian groups with
$$a \cdot n = \phi(a)n$$
as the multiplication by A.

Proposition 6.2. *There is a functorial bijection*

$$\begin{aligned} \mathrm{Hom}_B(M \otimes_A B, N) &\longleftrightarrow \mathrm{Hom}_A(M, N_A) \\ [f : M \otimes_A B \to N] &\longmapsto [m \mapsto f(m \otimes 1)] \\ [m \otimes b \mapsto b \cdot g(m)] &\longleftarrow [g : M \to N_A] \end{aligned}$$

for $M \in \mathrm{Mod}_A$ and $N \in \mathrm{Mod}_B$.

Remark. Given the result for rings, we want a similar statement for $\mathcal{O}_X$-modules.

6.2 $\mathcal{O}_X$-Modules and Continuous Maps

Definition 6.1. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the data of

1. a continuous map $f : X \to Y$,
2. a morphism of sheaves of rings $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Example 6.1.1. If $X \to Y$ is a morphism of varieties, then $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is given for $U \subseteq Y$ open by

$$\begin{aligned} \mathcal{O}_Y(U) &\longrightarrow \mathcal{O}_X(f^{-1}(U)) \\ \varphi &\longmapsto f^*\varphi. \end{aligned}$$

Remark. Our goal will be to define functors

$$\begin{tikzcd} \mathrm{Mod}_{\mathcal{O}_X} \arrow[r, bend left, "f_*"] & \mathrm{Mod}_{\mathcal{O}_Y} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Remark (Pushforward)**.** Given an $\mathcal{O}_X$-module $\mathcal{F}$, the sheaf pushforward $f_*\mathcal{F}$ is naturally an $f_*\mathcal{O}_X$-module. Via the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, we get an $\mathcal{O}_Y$-module structure on $f_*\mathcal{F}$. More concretely, for $U \subseteq Y$ open, $s \in f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, and $a \in \mathcal{O}_Y(U)$, we can define

$$a \cdot s = f^\#(U)(a) \cdot s.$$

Remark (Pullback)**.** Given an $\mathcal{O}_Y$-module $\mathcal{G}$, we get an $f^{-1}\mathcal{O}_Y$-module $f^{-1}\mathcal{G}$. By the adjoint property for (f^{-1}, f_*), the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ corresponds to a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. So we get

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

is an $\mathcal{O}_X$-module. Thus we get a functor $f^* : \mathrm{Mod}_{\mathcal{O}_Y} \to \mathrm{Mod}_{\mathcal{O}_X}$.

Proposition 6.3. *The pair (f^*, f_*) are adjoint functors.*

Proof. Similar to before. □

Example 6.1.2. Recall that if $A \to B$ is a morphism of rings, then $A \otimes_A B \cong B$. In our setting, we get

$$f^*\mathcal{O}_Y = (f^{-1}\mathcal{O}_Y \mathbin{\widetilde{\otimes}}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)^+ \cong \mathcal{O}_X^+ \cong \mathcal{O}_X.$$

Similarly, we have $f^*(\mathcal{O}_Y^{\oplus I}) \cong (f^*\mathcal{O}_Y)^{\oplus I} \cong \mathcal{O}_X^{\oplus I}$ (as left-adjoint functors commute with coproducts).

Remark. If $\mathcal{E}$ is a locally free rank m $\mathcal{O}_X$-module, then $f^*\mathcal{E}$ is a locally free rank m $\mathcal{O}_Y$-module, (as f^* can be computed locally on Y using Example 6.1.2).

Lecture 7

Feb. 3 — Coherent Sheaves

7.1 Review of Localization

Remark. Let A be a ring and $S \subseteq A$ a multiplicative system (i.e. $1 \in S$, and $a, b \in S$ implies $ab \in S$). For example, we could take $S = \langle f \rangle = (1, f, f^2, \dots)$ for $f \in A$ or $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p} \leq A$.

Definition 7.1. The *localization* of A at S is

$$S^{-1}A = \{a/s : a \in A, s \in S\},$$

where $a/s = a'/s'$ if and only if $t(as' - a's) = 0$ for some $t \in S$.

Remark. The localization satisfies the following universal property:

$$\begin{tikzcd} A \arrow[r, "a \mapsto a/1"] \arrow[dr, "f"'] & S^{-1}A \arrow[d, dashed, "\exists!"] \\ & T \end{tikzcd}$$

whenever $f(S)$ lands in the units of T.

Definition 7.2. For an A-module M, the *localization* of M at S is

$$S^{-1}M = \{m/s : m \in M, s \in S\},$$

where $m/s = m'/s'$ if and only if $t(s'm - sm') = 0$ for some $t \in S$.

Remark. For $S = \langle f \rangle$, we will write $S^{-1}M = M_f$. For $S = A \setminus \mathfrak{p}$, we will write $S^{-1}M = M_{\mathfrak{p}}$.

Proposition 7.1. *We have the following properties for localization:*

1. *There is an isomorphism*

$$\begin{aligned} M \otimes_A S^{-1}A &\xrightarrow{\cong} S^{-1}M \\ m \otimes (a/s) &\longmapsto (am)/s. \end{aligned}$$

2. *Localization gives an exact functor*

$$\begin{aligned} \mathrm{Mod}_A &\longrightarrow \mathrm{Mod}_{S^{-1}A} \\ M &\longmapsto S^{-1}M. \end{aligned}$$

3. *A sequence in* Mod_A
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
is exact if and only if the sequence
$$0 \longrightarrow M'_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}} \longrightarrow 0$$
is exact for all maximal (equivalently, prime) ideals $\mathfrak{p} \leq A$.

Example 7.2.1. Recall that if X is an affine variety, then
$$\mathcal{O}_X(D(f)) \cong A(X)_f \quad \text{and} \quad \mathcal{O}_{X,x} \cong A(X)_{\mathfrak{m}_x},$$
where $\mathfrak{m}_x = I(\{x\}) \leq A(X)$.

7.2 Coherent Sheaves on Affine Varieties

Remark. For this section, let X be an affine variety and $A = \mathcal{O}_X(X) = A(X)$. We want a functor
$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathcal{O}_X}.$$

Theorem 7.1. *For* $M \in \mathrm{Mod}_A$*, there exists a unique* $\mathcal{O}_X$*-module* $\widetilde{M}$ *such that*

1. $\widetilde{M}(D(f)) \cong M_f$;
2. *for* $D(g) \subseteq D(f)$*, we have*
$$\begin{tikzcd} \widetilde{M}(D(f)) \arrow[r] \arrow[d, "="'] & \widetilde{M}(D(g)) \arrow[d, "="] \\ M_f \arrow[r, "\text{natural map}"'] & M_g \end{tikzcd}$$

Remark. How is the natural map $M_f \to M_g$ defined? If $D(g) \subseteq D(f)$, then we have $V(g) \supseteq V(f)$, so $\sqrt{(g)} \subseteq \sqrt{(f)}$. Thus $g \in \sqrt{(f)}$, so $g^d = fh$ for some $d > 0$ and $h \in A$. So we get a map
$$\begin{aligned} M_f &\longrightarrow M_g \\ m/f^i &\longmapsto mh^i/g^{di}. \end{aligned}$$

Alternatively, if $D(g) \subseteq D(f)$, then $D(g) = D(gf)$, so we could instead consider
$$\begin{tikzcd} \widetilde{M}(D(f)) \arrow[r] \arrow[d, "="'] & \widetilde{M}(D(fg)) \arrow[d] \\ M_f \arrow[r, "m/f^i \mapsto mg^i/(fg)^i"'] & M_{fg} \end{tikzcd}$$

Remark. To construct $\widetilde{M}$, we need the notion of *sheaves on a basis*. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{P}$ a collection of open sets in X such that

1. $\mathcal{P}$ is a basis for X;

2. if $U, V \in \mathcal{P}$, then $U \cap V \in \mathcal{P}$.

Example 7.2.2. If $(X, \mathcal{O}_X)$ is an affine variety, we can take $\mathcal{P} = \{D(f) : f \in A(X)\}$. Also, if $(X, \mathcal{O}_X)$ is a general algebraic variety, then we can take $\mathcal{P}$ to be the affine open subsets of X (as X is separated, the intersection of two affine open subsets is again affine open).

Definition 7.3. A *$\mathcal{P}$-sheaf of $\mathcal{O}_X$-modules* $\mathcal{F}$ on X is the data of:

- an $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ for each $U \in \mathcal{P}$,
- homomorphisms of abelian groups $\mathcal{F}(U) \to \mathcal{F}(V)$ for all $U, V \in \mathcal{P}$ with $V \subseteq U$

satisfying the following properties:

- the multiplication is compatible with restriction,
- the sheaf axiom with respect to open sets in $\mathcal{P}$.

Example 7.3.1. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we get $\mathcal{F}^{\mathcal{P}}$, a $\mathcal{P}$-sheaf of $\mathcal{O}_X$-modules.

Theorem 7.2. *There is an equivalence of categories*

$$\begin{aligned} \mathrm{Mod}_{\mathcal{O}_X} &\longrightarrow \mathrm{Mod}_{\mathcal{O}_X}^{\mathcal{P}} \\ \mathcal{F} &\longmapsto \mathcal{F}^{\mathcal{P}}. \end{aligned}$$

In particular, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}^{\mathcal{P}}, \mathcal{G}^{\mathcal{P}})$ *is a bijection, and for any* $\mathcal{H} \in \mathrm{Mod}_{\mathcal{O}_X}^{\mathcal{P}}$*, there exists some* $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ *such that* $\mathcal{F}^{\mathcal{P}} \cong \mathcal{H}$*.*

Proof. We construct the inverse functor. Take $\mathcal{H} \in \mathrm{Mod}_{\mathcal{O}_X}^{\mathcal{P}}$, and define $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ by setting

$$\mathcal{F}(U) = \{(s_V)_{V \in \mathcal{P}, V \subseteq U} : s_V|_{V \cap V'} = s'_V|_{V \cap V'} \text{ for all } V, V' \in \mathcal{P} \text{ with } V, V' \subseteq U\}.$$

One can check that this defines a functor and an equivalence of categories. □

Remark. Returning to algebraic geometry, let X be an affine variety, $A = A(X)$, and M an A-module.

Proposition 7.2. *For* $\mathcal{P} = \{D(f) : f \in A\}$*, the assignment* $D(f) \mapsto M_f$ *with the natural restriction maps defines a* $\mathcal{P}$*-sheaf of* $\mathcal{O}_X$*-modules.*

Proof. See Mustata Lemma 8.3.2. The hard part is to check the sheaf axiom for $\mathcal{P}$, which is similar to the computation that $\mathcal{O}_X(D(f)) \cong A_f$ for an affine variety X. □

Proof of Theorem 7.1. Combining Theorem 7.2 and Proposition 7.2, we get an $\mathcal{O}_X$-module $\widetilde{M}$ such that $\widetilde{M}(D(f)) \cong M_f$, and it is unique up to isomorphism. □

Example 7.3.2. We have $\widetilde{A} \cong \mathcal{O}_X$, since

$$\widetilde{A}(D(f)) \cong A_f \cong \mathcal{O}_X(D(f)).$$

Similarly, one can check that $\widetilde{A^{\oplus I}} \cong \mathcal{O}_X^{\oplus I}$.

Exercise 7.1. For $Z \subseteq X$ closed and $I(Z) \leq A(X)$, we have $\widetilde{I(Z)} \cong \mathcal{I}_Z$ (the ideal sheaf of Z).

Remark. We have the following:

1. For $x \in X$ with $\mathfrak{p} := I(\{x\}) \leq A$, we have $\widetilde{M}_x \cong \varinjlim_{D(f)\ni x} \widetilde{M}(D(f)) = \varinjlim_{f \in A \setminus \mathfrak{p}} M_f \cong M_{\mathfrak{p}}$.
2. For an A-module homomorphism $\varphi : M \to N$, we get homomorphisms

$$\begin{tikzcd} \widetilde{M}(D(f)) \arrow[r] \arrow[d, "\cong"'] & \widetilde{N}(D(f)) \arrow[d, "\cong"] \\ M_f \arrow[r, "\text{natural map}"'] & N_f \end{tikzcd}$$

which are compatible with restriction. So we get a homomorphism of $\mathcal{O}_X$-modules $\widetilde{M} \to \widetilde{N}$. One can check that this gives a functor $\Phi : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}$.

Proposition 7.3. *We have the following:*

1. *Φ is exact;*
2. *Φ is fully faithful, i.e. the following map is a bijection:*

$$\begin{aligned} \mathrm{Hom}_A(M, N) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \\ \varphi &\longmapsto \widetilde{\varphi}. \end{aligned}$$

Proof. (1) If $0 \to M' \to M \to M'' \to 0$ is exact, then $0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0$ is exact for every $\mathfrak{p} \leq A$ maximal by Proposition 7.1(3). So the sequence

$$0 \longrightarrow \widetilde{M}'_x \longrightarrow \widetilde{M}_x \longrightarrow \widetilde{M}''_x \longrightarrow 0$$

is exact for all $x \in X$. So $0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$ is exact.

(2) We want a map $\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_A(M, N)$. Given an $\mathcal{O}_X$-module homomorphism $\varphi : \widetilde{M} \to \widetilde{N}$, we get $f = \varphi(X) : \widetilde{M}(X) = M \to \widetilde{N}(X) = N$. We want to show that $\widetilde{f} = \varphi$. We have

$$\begin{tikzcd} \widetilde{M}(X) \arrow[r, "\varphi(X)"] \arrow[d] & \widetilde{N}(X) \arrow[d] \\ \widetilde{M}(D(g)) \arrow[r, "\varphi(D(f))"'] & \widetilde{N}(D(g)) \end{tikzcd}$$

and this diagram commutes. So we get $\widetilde{f}(D(g)) = \varphi(D(g))$. $\square$

Remark. By Proposition 7.3(1), given an A-module homomorphism $f : M \to N$, we have

$$\ker(\widetilde{f}) \cong \widetilde{\ker(f)}, \quad \mathrm{coker}(\widetilde{f}) \cong \widetilde{\mathrm{coker}(f)}, \quad \mathrm{im}(\widetilde{f}) \cong \widetilde{\mathrm{im}(f)}.$$

Proposition 7.4. *For $M, N \in \mathrm{Mod}_A$, we have*

1. *$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_A N}$;*
2. *$\widetilde{\mathrm{Hom}_A(M, N)} \cong \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$;*
3. *$\widetilde{\bigoplus_{i\in I} M_i} \cong \bigoplus_{i \in I} \widetilde{M}_i$.*

Proof. (1) We have a homomorphism

$$\begin{tikzcd} M \otimes_A N \arrow[r, "\cong"] & \Gamma(X, \widetilde{M} \widetilde{\otimes}_{\mathcal{O}_X} \widetilde{N}) \arrow[d] \\ & \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \end{tikzcd}$$

This gives a homomorphism of $\mathcal{O}_X$-modules $\widetilde{M \otimes_A N} \to \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. Now at the stalks for $x \in X$ with $\mathfrak{m} = I(\{x\}) \leq A(X)$, we can see that

$$(\widetilde{M \otimes_A N})_x \cong (M \otimes_A N) \otimes_A A_{\mathfrak{m}} \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Similarly, we have

$$(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})_x \cong \widetilde{M}_x \otimes_{\mathcal{O}_{X,x}} \widetilde{N}_x \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Thus we have isomorphisms at the stalks. □

Remark. The functor $\Phi : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}$ is left adjoint to $\mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_A$ given by $\mathcal{F} \mapsto \mathcal{F}(X)$.

Lecture 8

Feb. 5 — Coherent Sheaves, Part 2

8.1 Coherent Sheaves on Affine Varieties, Continued

Proposition 8.1. *Let $\varphi : X \to Y$ be a morphism of affine varieties, and let*

$$A = \mathcal{O}_Y(Y) \xrightarrow{\varphi^\#} \mathcal{O}_X(X) = B$$

be the corresponding ring homomorphism on the coordinate rings. Then:

1. *For $N \in \mathrm{Mod}_B$, we have $\varphi_* \widetilde{N} = \widetilde{N_A}$.*
2. *For $M \in \mathrm{Mod}_A$, we have $\varphi^* \widetilde{M} = \widetilde{M \otimes_A B}$.*

Proof. (1) For $f \in A = \mathcal{O}_Y(Y)$, the left-hand side is given on $D(f)$ by

$$(\varphi_* \widetilde{N})(D(f)) = \widetilde{N}(\varphi^{-1}(D(f))) = \widetilde{N}(D(\varphi^\# f)) = N_{\varphi^\# f} = (N_A)_f.$$

So we get that $\varphi_* \widetilde{N} = \widetilde{N_A}$.

(2) (Hartshorne says this holds by definition lol.) First assume $M = A^{\oplus I}$. Then

$$\varphi^* \widetilde{M} = \varphi^*(\mathcal{O}_Y^{\oplus I}) = (\varphi^* \mathcal{O}_Y)^{\oplus I} = \mathcal{O}_X^{\oplus I} = \widetilde{B^{\oplus I}} = \widetilde{M \otimes_A B}.$$

For an arbitrary A-module M, we use the following:

> **Claim.** There exists an exact sequence
>
> $$A^{\oplus J} \xrightarrow{\alpha} A^{\oplus I} \xrightarrow{\beta} M \longrightarrow 0$$
>
> *Proof of claim.* Choose generators $(m_i)_{i \in I}$ for M, and set $\beta(e_i) = m_i$. Choose generators $(n_j)_{j \in J}$ for $\ker \beta$. Then we can set $\alpha(f_j) = n_j$. □

Apply $\cdot \otimes_A B$ (which is right exact) to the exact sequence from the claim to get

$$B^{\oplus J} \longrightarrow B^{\oplus I} \longrightarrow M \otimes_A B \longrightarrow 0.$$

Applying $\varphi^*(\widetilde{\cdot})$ (note that $\widetilde{\cdot}$ is exact and φ^* is right exact) to the original sequence to get

$$\begin{array}{ccccccc} \varphi^*(\widetilde{A^{\oplus J}}) & \longrightarrow & \varphi^*(\widetilde{A^{\oplus I}}) & \longrightarrow & \varphi^* \widetilde{M} & \longrightarrow & 0 \\ \| & & \| & & & & \\ B^{\oplus J} & & B^{\oplus I} & & & & \end{array}$$

using the previous case. Thus $\varphi^* \widetilde{M} = \mathrm{coker}(\widetilde{B^{\oplus J}} \to \widetilde{B^{\oplus I}}) = (\mathrm{coker}(B^{\oplus J} \to B^{\oplus I}))^\sim = \widetilde{M \otimes_A B}$. □

8.2 Quasicoherent and Coherent Sheaves

Remark. For the rest of this lecture, assume $(X, \mathcal{O}_X)$ is a variety (not necessarily affine).

Definition 8.1. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasicoherent* if there exists an affine cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some $\mathcal{O}_X(U_i)$-module M_i. It is *coherent* if the M_i are finitely generated $\mathcal{O}_X(U_i)$-modules.

Remark. We have $M_i \cong \widetilde{M_i}(U_i) \cong \mathcal{F}(U_i)$, so we may replace the $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ condition with one of the following: $\mathcal{F}|_{U_i} \cong \widetilde{\mathcal{F}(U_i)}$, or $\mathcal{F}(U_i)_f \to \mathcal{F}(D_{U_i}(f))$ is an isomorphism for all $f \in \mathcal{O}_X(U_i)$.

Example 8.1.1. We have the following:

1. If $\mathcal{E}$ is a locally free sheaf of rank r on X, then there exists an open cover $X = \bigcup U_i$ such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_X^{\oplus r}|_{U_i}$. Refining the cover, we may assume the U_i are affine, so

$$\mathcal{E}|_{U_i} = \widetilde{\mathcal{O}_X(U_i)^{\oplus r}}.$$

   Thus we see that $\mathcal{E}$ is coherent.

2. For $Z \hookrightarrow X$ a closed embedding, $\mathcal{I}_Z \subseteq \mathcal{O}_X$ is coherent.

3. For $\mathcal{F}$ a (quasi)coherent $\mathcal{O}_X$-module and $U \subseteq X$ open, $\mathcal{F}|_U$ is (quasi)coherent.

   To see this, use that if $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for M_i an $\mathcal{O}_X(U_i)$-module and $f \in \mathcal{O}_X(U_i)$, then

$$\mathcal{F}|_{D_{U_i}(f)} \cong \widetilde{(M_i)_f}.$$

   Furthermore, if M_i is finitely generated, then so is $(M_i)_f$. So by refining our open affine cover in with principal opens (which form a basis) we may assume $U = \bigcup_{i, U_i \subseteq U} U_i$. This gives the result.

Proposition 8.2 (Key proposition)**.** *Let $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$. The following are equivalent:*

1. *$\mathcal{F}$ is quasicoherent (resp. coherent).*
2. *For any affine open set $U \subseteq X$, we have $\mathcal{F}|_U \cong \widetilde{\mathcal{F}(U)}$ (with $\mathcal{F}(U)$ finitely generated in the coherent case).*

Lemma 8.1 (Affine communication)**.** *If X is a variety, $U, V \subseteq X$ affine open subsets, and $p \in U \cap V$, then there exists an open set $p \in W \in U \cap V$ that is a principal open of both U and V.*

Proof. Choose a principal open of U with $p \in W_1 = D_U(h) \subseteq U \cap V$ for some $h \in \mathcal{O}_X(U)$, and choose a principal open of V with $p \in W = D_V(g) \subseteq W_1$ for some $g \in \mathcal{O}_X(V)$. Now

$$g|_{W_1} = \mathcal{O}_X(W_1) = \mathcal{O}_X(U)_h,$$

so $g|_{W_1} = f/h^i$ for some $f \in \mathcal{O}_X(U)$ and $i \geq 0$. Now $D_V(g) = W = D_{W_1}(g|_{W_1}) = D_U(fh)$. $\square$

Proof of Proposition 8.2. $(2 \Rightarrow 1)$ This is clear.

$(1 \Rightarrow 2)$ Assume $\mathcal{F} \in \mathrm{QCoh}_X$. So there exists an open affine cover $\{U_i\}$ such that $\mathcal{F}|_{U_i} = \widetilde{\mathcal{F}(U_i)}$. Fix $U \subseteq X$ affine open. By refining the cover, we may assume that $U = \bigcup_{U_i \subseteq U} U_i$. Now replacing X with U, we may assume that $X = U$. Using Lemma 8.1, we may assume $U_i = D(f_i)$ for some $f_i \in \mathcal{O}_X(X)$.

So now X is affine, $X = \bigcup_{i=1}^r D(f_i)$, and $\mathcal{F}|_{D(f_i)} \cong \widetilde{\mathcal{F}(D(f_i))}$. We want to show that for any $f \in A$, the natural map $\mathcal{F}(X)_f \to \mathcal{F}(D(f))$ is an isomorphism (this would imply $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$). Now

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X)_f \arrow[r] \arrow[d, "\alpha"] & \bigoplus_i \mathcal{F}(D(f_i))_f \arrow[r] \arrow[d, "\beta"] & \bigoplus_{i,j} \mathcal{F}(D(f_i f_j))_f \arrow[d, "\gamma"] \\
0 \arrow[r] & \mathcal{F}(D(f)) \arrow[r] & \bigoplus_i \mathcal{F}(D(f_i f)) \arrow[r] & \bigoplus_{i,j} \mathcal{F}(D(f_i f_j f))
\end{tikzcd}$$

where the first row is exact by the sheaf property and using that localization is exact, and the second row is exact by the sheaf property. Note that β and γ are isomorphisms as

$$\mathcal{F}|_{D(f_i)} \cong \widetilde{\mathcal{F}(D(f_i))} \quad \text{and} \quad \mathcal{F}|_{D(f_i f_j)} \cong \widetilde{\mathcal{F}(D(f_i f_j))}.$$

So by the five lemma, α is an isomorphism, and thus $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$. □

Remark. For the coherent case, use the fact that if M is an A-module, $A = (f_1, \ldots, f_r)$, and M_{f_i} is finitely generated for $i = 1, \ldots, r$, then M is finitely generated.

Proposition 8.3. *We have the following:*

1. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of (quasi)coherent sheaves, then $\ker\varphi, \operatorname{im}\varphi, \operatorname{coker}\varphi$ are (quasi)coherent.*
2. *If $\mathcal{F}$ and $\mathcal{G}$ are (quasi)coherent, then so are $\mathcal{F} \otimes \mathcal{G}$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.*
3. *If $\mathcal{F}_i$ is quasicoherent for $i \in I$, then $\bigoplus_{i \in I} \mathcal{F}_i$ is quasicoherent. Furthermore, if $\mathcal{F}_i$ is coherent and $|I| < \infty$, then $\bigoplus_{i \in I} \mathcal{F}_i$ is coherent.*

Proof. (1) Choose $U \subseteq X$ affine open. So we can write $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \widetilde{N}$. Furthermore, $\varphi|_U = \widetilde{\alpha}$ for some $\mathcal{O}_X(U)$-module homomorphism $\alpha : M \to N$. Now

$$(\ker\varphi)|_U = \ker\varphi|_U = \ker\widetilde{\alpha} = \widetilde{\ker\alpha}.$$

Furthermore, if $\mathcal{F}$ and $\mathcal{G}$ are coherent, then M and N are finitely generated, so $\ker\alpha$ is also finitely generated. One can show the same for $\operatorname{im}\varphi$ and $\operatorname{coker}\varphi$ similarly. □

Remark. We have full subcategories $\mathrm{Coh}_X \subseteq \mathrm{QCoh}_X \subseteq \mathrm{Mod}_{\mathcal{O}_X}$, which are abelian.

Lecture 9

Feb. 10 — Coherent Sheaves, Part 3

9.1 Quasicoherent Sheaves, Continued

Proposition 9.1. *Let $f : X \to Y$ be a morphism of varieties. Then*

1. *If $\mathcal{G} \in \mathrm{QCoh}_Y$ (resp. Coh_Y), then $f^*\mathcal{G} \in \mathrm{QCoh}_X$ (resp. Coh_X).*
2. *If $\mathcal{F} \in \mathrm{QCoh}_X$, then $f_*\mathcal{F} \in \mathrm{QCoh}_Y$.*

Proof. (1) Fix $\mathcal{G} \in \mathrm{QCoh}_Y$ and $x \in X$. There exist affine opens $x \in U \subseteq X$ and $f(x) \in V \subseteq Y$ such that $f(U) \subseteq V$. Write $g = f|_U : U \to V$, then $(f^*\mathcal{G})|_U = g^*(\mathcal{G}|_V)$. Now letting $A = \mathcal{O}_Y(V)$, $B = \mathcal{O}_X(U)$, $M = \mathcal{G}(V)$, we see that

$$f^*\mathcal{G}|_U = g^*(\mathcal{G}|_V) = g^*\widetilde{M} = \widetilde{M \otimes_A B}.$$

So $f^*\mathcal{G}$ is quasicoherent. The coherent version follows from the fact that if M is a finitely generated A-module, then $M \otimes_A B$ is a finitely generated B-module.

(2) Fix $\mathcal{F} \in \mathrm{QCoh}_X$. We can check quasicoherence locally, so we can reduce to the case where Y is affine (cover Y by affine opens U_i and replace f with $f^{-1}(U_i) \to U_i$). Choose an affine cover $X = U_1 \cup \cdots \cup U_r$, and note that $U_i \cap U_j$ is again affine for a variety. Write $\alpha_i : U_i \hookrightarrow X$ and $\alpha_{i,j} : U_{i,j} \hookrightarrow X$. As $\mathcal{F}$ is a sheaf, we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \textstyle\bigoplus_i (\alpha_i)_*\mathcal{F}|_{U_i} \longrightarrow \bigoplus_{i,j} (\alpha_{i,j})_*\mathcal{F}|_{U_{i,j}}.$$

Applying f_*, which is left exact, we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \textstyle\bigoplus_i (f \circ \alpha_i)_*\mathcal{F}|_{U_i} \longrightarrow \bigoplus_{i,j} (f \circ \alpha_{i,j})_*\mathcal{F}|_{U_{i,j}}.$$

Note that the last two terms are both quasicoherent (e.g. $\mathcal{F}|_{U_i}$ is quasicoherent and $f \circ \alpha_i$ is a morphism of affine varieties, so $(f \circ \alpha_i)_*\mathcal{F}|_{U_i}$ is quasicoherent).[1] □

Remark. The coherent version of Proposition 9.1(2) fails: For

$$i : \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1,$$

the pushforward $i_*\mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$ is not coherent (but $\mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$ is coherent).

[1] If $f : X \to Y$ is a morphism of affine varieties, and $A := \mathcal{O}_Y(Y)$, $B := \mathcal{O}_X(X)$, then $f_*\widetilde{N} = \widetilde{N_A}$ for $N \in \mathrm{Mod}_B$ and $f^*\widetilde{M}$ for $M \in \mathrm{Mod}_A$. This shows that pushforwards and pullbacks of quasicoherent sheaves on *affine* varieties are again quasicoherent.

Remark. If $X \to Y$ is *projective*, i.e. there exists a factorization

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & Y \times \mathbb{P}^n \arrow[d, "\mathrm{pr}_1"] \\ & Y \end{tikzcd}$$

then we do get a coherent version for the pushfoward $f_*\mathcal{F}$. One can prove this via sheaf cohomology.

9.2 Morphisms to Projective Space

Remark. Recall that there is a bijection

$$\{\mathbb{L} \to X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\}/\cong \longleftrightarrow \{\text{morphisms } X \to \mathbb{P}^n\}.$$

We want to rephrase this using the bijection

$$\{\mathcal{L} \text{ invertible sheaves of } \mathcal{O}_X\text{-modules}\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \to X\}$$

and determine when $X \to \mathbb{P}^n$ is injective or a closed embedding.

Remark. Let X be a variety and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. For $x \in X$, we have

$$\mathcal{L}(x) := \mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x \cong \mathcal{O}_{X,x}/\mathfrak{m}_x \cong k,$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is a maximal ideal.

Definition 9.1. We say $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ *generate* $\mathcal{L}$ if

$$s_0(x), \dots, s_n(x) \in \mathcal{L}(x) \cong k$$

generate $\mathcal{L}(x)$ as a k-vector space (i.e. at least one of $s_0(x), \dots, s_n(x)$ is nonzero) for all $x \in X$.

Proposition 9.2. *The following are equivalent:*

1. $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ *generate* $\mathcal{L}$*;*
2. $(s_0)_x, \dots, (s_n)_x$ *generate* $\mathcal{L}_x$ *as an* $\mathcal{O}_{X,x}$*-module for all* $x \in X$*;*
3. *the morphism* $\varphi : \mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}$ *given by* $e_i \mapsto s_i$ *is surjective;*
4. *for every* $U \subseteq X$ *open affine,* $s_0|_U, \dots, s_n|_U$ *generate* $\mathcal{L}(U)$ *as an* $\mathcal{O}_X(U)$*-module.*

Proof. By commutative algebra (in particular Nakayama's lemma), $s_0(x), \dots, s_n(x)$ span $\mathcal{L}(x)$ (1) if and only if $(s_0)_x, \dots, (s_n)_x$ generate $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module (2). This happens if and only if φ_x is surjective, which happens if and only if φ is surjective (3). This happens if and only if $\varphi(U)$ is surjective for all $U \subseteq X$ open affine (4) (note that $\operatorname{coker}(\widetilde{M} \to \widetilde{N}) = (\operatorname{coker}(M \to N))^\sim$). $\square$

Lecture 10

Feb. 12 — Morphisms to Projective Space

10.1 Morphisms to Projective Space, Continued

Definition 10.1. When any of the conditions in Proposition 9.2 hold, we say that $\mathcal{L}$ is *generated* by $s_0, \ldots, s_n$. Furthermore, we say $\mathcal{L}$ is *globally generated* if there exist $s_0, \ldots, s_n$ generating $\mathcal{L}$.

Example 10.1.1. $\mathcal{O}_X$ is globally generated, e.g. by $1 \in \Gamma(X, \mathcal{O}_X)$.

Example 10.1.2. For $X = \mathbb{P}^n$, the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ defined by transition data $(U_i, x_j/x_i)$ is globally generated. There are isomorphisms $\alpha_i : \mathcal{O}_{\mathbb{P}^n}(1)|_{U_i} \to \mathcal{O}_{U_i}$ such that $\alpha_i \circ \alpha_j^{-1}$ = multiplication by x_j/x_i. Note that we have an isomorphism

$$\begin{aligned} k[x_0, \ldots, x_n]_1 &\longrightarrow \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \\ f &\longmapsto s_f, \end{aligned}$$

where $\alpha_i(s_f|_{U_i}) = f/x_i$ (check that this is well-defined).

Then $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ generate $\mathcal{O}_{\mathbb{P}^n}(1)$. To see this, note that on U_i, we have

$$\alpha_i(x_i) = 1 \in \mathcal{O}_{U_i},$$

so $x_i|_{U_i}$ generates $\mathcal{O}_{\mathbb{P}^n}(1)(U_i)$ as an $\mathcal{O}_{\mathbb{P}^n}(U_i)$-module by Proposition 9.2(4). So the $x_0, \ldots, x_n$ generate $\mathcal{O}_{\mathbb{P}^n}(1)$. Alternatively, one can note that for $a = [a_0 : \cdots : a_n] \in \mathbb{P}^n$, we have $[x_0(a) : \cdots : x_n(a)] = [a_0 : \cdots : a_n]$, so the $x_i(a)$ do not all vanish, hence the x_i generate $\mathcal{O}_{\mathbb{P}^n}(1)$.

Example 10.1.3. Let $f : X \to \mathbb{P}^n$ be a morphism. Then $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by

$$f^*x_0, \ldots, f^*x_n \in \Gamma(X, \mathcal{L}).$$

Remark. If $\mathcal{L}$ is generated by $s_0, \ldots, s_n$, then we get a map

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \cdots : s_n(x)]. \end{aligned}$$

This is a morphism: If we fix $a \in X$ and pick $a \in U \subseteq X$ open such that

$$\alpha : \mathcal{L}|_U \xrightarrow{\cong} \mathcal{O}_U,$$

then $f|_U(x) = [\alpha(s_0|_U)(x) : \cdots : \alpha(s_n|_U)(x)]$. Since each $\alpha(s_i)$ is a regular function, $f|_U$ is a morphism. Thus we see that f is a morphism.

Remark. Similar to before, we have a bijection

$$\{\text{morphisms } X \to \mathbb{P}^n\} \longleftrightarrow \{\text{invertible sheaves } \mathcal{L} \text{ on } X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathcal{L}) \text{ generators}\}/\cong$$

The maps are given by $f \mapsto \mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ with $s_i = f^*x_i$ with inverse $(\mathcal{L}, s_i) \mapsto [x \mapsto [s(x)]]$.

Example 10.1.4. Recall the *Veronese embedding* given by

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^{Nd} \\ a &\longmapsto [x^I(a) : x^I \text{ monomial of degree } d \text{ in } x_0, \dots, x_n]. \end{aligned}$$

This map corresponds to $\mathcal{O}_{\mathbb{P}^n}(d)$ with sections $x^I \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$.

10.2 Injectivity and Closed Embeddings

Remark. When is $f : X \to \mathbb{P}^n$ injective or a closed embedding?

Definition 10.2. Let $V = \operatorname{span}\{s_0, \dots, s_n\} \subseteq \Gamma(X, \mathcal{L})$ such that $s_0, \dots, s_n$ generate $\mathcal{L}$. We say that

1. $s_0, \dots, s_n$ *separate points* of X if for all $x \neq y$, there exists $s \in V$ such that $s(x) = 0 \neq s(y)$.[1]
2. $s_0, \dots, s_n$ *separate tangent vectors* if for each $p \in X$,
$$\{s_p : s \in V \text{ and } s(p) = 0\}$$
generates $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$ as a k-vector space.[2]

Proposition 10.1. *Assume that X is complete and $\mathcal{L}$ an invertible sheaf on X generated by $s_0, \dots, s_n \in \Gamma(X, Y)$. Write*

$$\begin{aligned} f : X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \cdots : s_n(x)]. \end{aligned}$$

Then we have the following:

1. *f is injective if and only if $s_0, \dots, s_n$ separate points;*
2. *f is a closed embedding if and only if $s_0, \dots, s_n$ separate points and tangent vectors.*

Proof. (1) For $s = \sum_{i=0}^n a_i s_i$ with $a_i \in k$, we have

$$\{x \in X : s(x) = 0\} = f^{-1}(H_a)$$

where $H_a = V(\sum a_i x_i)$. Now f is injective if and only if for all $x \neq y \in X$, there exists a hyperplane $H \subseteq \mathbb{P}^n$ such that $f(x) \in H$ and $f(y) \notin H$. This happens if and only if $s_0, \dots, s_n$ separate points.

(2) Assume that f is injective (equivalently, $s_0, \dots, s_n$ separate points by (1)). For $p \in X$, we get a local ring homomorphism $\mathcal{O}_{\mathbb{P}^n, f(p)} \to \mathcal{O}_{X,p}$, which induces a map on cotangent spaces

$$\begin{tikzcd} \mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 \arrow[r] \arrow[d, "\cong"'] & \mathfrak{m}_p/\mathfrak{m}_p^2 \arrow[d, "\cong"] \\ \mathfrak{m}_{f(p)}\mathcal{O}_{\mathbb{P}^n}(1)_{f(p)}/\mathfrak{m}_{f(p)}^2\mathcal{O}_{\mathbb{P}^n}(1)_{f(p)} & \mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p \end{tikzcd}$$

[1] Note that $s(p) = 0$ if and only if $s \in \mathfrak{m}_p\mathcal{L}_p$.
[2] Here $\mathfrak{m}_p \leq \mathcal{O}_{X,p}$ is its unique maximal ideal. Recall that $(T_pX)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2 \cong \mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$.

(as $\mathcal{L}$ is invertible, it is locally trivial), where the bottom map is given by

$$\left[\sum a_i x_i\right] \longmapsto \left[\sum a_i s_i\right].$$

So f separates tangent vectors at p if and only if $\mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$ is surjective. Since $f : X \hookrightarrow \mathbb{P}^n$ is injective and a closed map (as X is complete), we get $X \hookrightarrow f(X) \subseteq \mathbb{P}^n$ is a homeomorphism (we know it is a bijection, preimages of open sets are open, and images of closed sets are closed), and $f(X) \subseteq \mathbb{P}^n$ is closed. Using this, f is a closed embedding if and only if $\mathcal{O}_{\mathbb{P}^n} \to f_*\mathcal{O}_X$ is surjective (HW) which happens if and only if $\mathcal{O}_{\mathbb{P}^n,f(p)} \to \mathcal{O}_{X,p}$ is surjective for all $p \in X$. This happens if and only if

$$\mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 \longrightarrow \mathfrak{m}_p/\mathfrak{m}_p^2$$

is surjective for all $p \in X$ (see Mustata's notes, uses that $f_*\mathcal{O}_X$ is coherent). □

Definition 10.3. Let $\mathcal{L}$ be an invertible sheaf on a complete variety X.

1. $\mathcal{L}$ is *very ample* if there exists a closed embedding $f : X \hookrightarrow \mathbb{P}^n$ and $\mathcal{L} \cong \mathcal{O}_X(1) := f^*\mathcal{O}_{\mathbb{P}^n}(1)$.
2. $\mathcal{L}$ is *ample* if there exists $m > 0$ such that $\mathcal{L}^{\otimes m}$ is very ample.

Example 10.3.1. Let $X = \mathbb{P}^n$ (for $n \geq 1$), then $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ is very ample if and only if $d > 0$, if and only if $\mathcal{O}_{\mathbb{P}^n}(d)$ is ample (though this is not true in general).

Exercise 10.1. If $\mathcal{F}$ is a coherent sheaf on a complete variety X and $\mathcal{O}_X(1)$ is ample, then show that there exists an exact sequence:

$$\bigoplus_{i=1}^s \mathcal{O}_X(n_i) \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_X(m_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

10.3 Divisors

Remark. For the rest of the lecture, let X be an irreducible variety.

Definition 10.4. A *prime divisor* on X is a closed irreducible subvariety $D \subseteq X$ of codimension 1.

Example 10.4.1. We have the following:

1. Prime divisors on $\mathbb{P}^n$ correspond to irreducible hypersurfaces.
2. Prime divisors on a curve C correspond to points.

Definition 10.5. A *(Weil) divisor* on X is a formal sum

$$D = \sum_{i=1}^r a_i D_i,$$

such $a_i \in \mathbb{Z}$ and the D_i are distinct prime divisors. Write

$$\operatorname{Div} X = \{\text{divisors on } X\} = \bigoplus_{\substack{E \subseteq X \\ \text{prime divisor}}} \mathbb{Z}[E].$$

Remark. We will see that when X is smooth, there is a surjective group homomorphism

$$\begin{aligned} \Phi : \operatorname{Div} X &\longrightarrow \operatorname{Pic} X = \{\text{invertible sheaves on } X\}/\cong \\ D &\longmapsto \mathcal{O}_X(D). \end{aligned}$$

Furthermore, $\ker \Phi = \{\operatorname{div} \varphi : \varphi \in K(X)^\times\}$, where $\operatorname{div} \varphi$ is the divisor of zeros and poles of φ.

Definition 10.6. For a closed irreducible subset $Z \subseteq X$, the *local ring at Z* is

$$\begin{aligned}\mathcal{O}_{X,Z} &= \{(\varphi, U) : \varphi \in \mathcal{O}_X(U),\, U \subseteq X \text{ open},\, Z \cap U \neq \varnothing\}/\sim \\ &= \varinjlim_{\substack{U \subseteq X \text{ open} \\ Z \cap U \neq \varnothing}} \mathcal{O}_X(U),\end{aligned}$$

where $(\varphi, U) \sim (\psi, V)$ if there exists $W \subseteq U \cap V$ open with $Z \cap W \neq \varnothing$ such that $\varphi|_W = \psi|_W$.

Remark. We consider open sets intersecting Z (and not those containing Z) because otherwise there not be many such U. For example, if $X = \mathbb{P}^n$ and Z is a hyperplane, then we can only have $U = X$.

Remark. This construction works for any sheaf on X, not just $\mathcal{O}_X$.

Lecture 11

Feb. 17 — Divisors

11.1 Divisors, Continued

Proposition 11.1. *We have the following:*

1. *If $U \subseteq X$ is affine open, then we have an isomorphism*
$$\mathcal{O}_X(U)_{\mathcal{I}_Z(U)} \xrightarrow{\cong} \mathcal{O}_{X,Z}$$
when $Z \cap U \neq \varnothing$.
2. $\dim \mathcal{O}_{X,Z} = \operatorname{codim}_X Z$.
3. *$\mathcal{O}_{X,Z}$ is a local ring.*
4. *Let $\mathfrak{m}_Z$ be the maximal ideal of $\mathcal{O}_{X,Z}$. If X is smooth, then $\mathcal{O}_{X,Z}$ is regular, i.e.*
$$\dim \mathcal{O}_{X,Z} = \dim \mathfrak{m}_Z/\mathfrak{m}_Z^2,$$
where the right-hand side is the dimension as a vector space over $\mathcal{O}_{X,Z}/\mathfrak{m}_Z$.

Proof. (1) The proof is the same as when $Z = \{x\}$.

(2) We can reduce to the case where X is affine. Then we have a bijection

$$\begin{array}{ccc} \{\text{prime ideals } \mathfrak{p} \leq A \text{ such that } \mathfrak{p} \cap (A \setminus I_Z) = \varnothing\} & \overset{\mathfrak{q} \mapsto \mathfrak{q}\mathcal{O}_{X,Z}}{\longleftrightarrow} & \{\text{prime ideals of } \mathcal{O}_{X,Z}\} \\ \updownarrow & & \\ \{\text{closed irreducible subvarieties } Z \subseteq Y \subseteq X\} & & \end{array}$$

where $A = \mathcal{O}_X(X)$. This proves the claim.

(3) We use (1). Since $\mathcal{I}_Z(U) \subseteq \mathcal{O}_X(U)$ is prime, $\mathcal{O}_{X,Z}$ is a local ring with maximal ideal

$$\mathfrak{m}_Z := \mathcal{I}_Z(U)\mathcal{O}_{X,Z} = \{(\varphi, U) \in \mathcal{O}_{X,Z} : \varphi|_{Z\cap U} = 0\} \subseteq \mathcal{O}_{X,Z}.$$

(4) In Algebraic Geometry I, we have seen that this holds when Z is a point. Choose $x \in Z$, then

$$\mathcal{O}_{X,Z} \cong (\mathcal{O}_{X,x})_{\{\varphi \in \mathcal{O}_{X,x} \,:\, \varphi|_Z = 0\}}.$$

As $\mathcal{O}_{X,x}$ is regular, so is $\mathcal{O}_{X,Z}$ by commutative algebra. $\square$

Remark. For the rest of this lecture, assume that X is a smooth irreducible variety (so that the local rings are regular and we can talk about the function field of X).

Remark. Let $E \subseteq X$ be a prime divisor. Then $\mathcal{O}_{X,E}$ is a regular local ring of dimension 1. By results from commutative algebra, $\mathcal{O}_{X,E}$ is a *discrete valuation ring* (DVR). In particular, this implies that $\mathcal{O}_{X,E}$ is an integral domain which is a UFD with a unique irreducible element up to multiplication by units.

So there exist an irreducible element $\pi \in \mathcal{O}_{X,E}$ such that for any $0 \neq f \in \operatorname{Frac}(\mathcal{O}_{X,E}) = K(X)$,

$$f = u\pi^d$$

for some $u \in \mathcal{O}_{X,E}^\times$ and $d \in \mathbb{Z}$ (with $d \geq 0$ if and only if $f \in \mathcal{O}_{X,E}$). Such a π is called a *uniformizer*.

We will write $\operatorname{ord}_E(f) = d$ when $f = u\pi^d$, which we view as the multiplicity of vanishing of f along E.

Remark. Note the following:

- $\operatorname{ord}_E(fg) = \operatorname{ord}_E(f) + \operatorname{ord}_E(g)$.
- $\operatorname{ord}_E(f + g) \geq \min\{\operatorname{ord}_E(f), \operatorname{ord}_E(g)\}$.

Along with some other properties, the above implies that ord_E is a *valuation* on $K(X)$.

Example 11.0.1. Consider $0 \in \mathbb{A}^1$. Then we have

$$\mathcal{O}_{\mathbb{A}^1,0} \cong \left\{ \frac{f}{g} : f, g \in k[x],\ g(0) \neq 0 \right\}.$$

For $\varphi \in \mathcal{O}_{\mathbb{A}^1,0}$, we can write

$$\varphi = ux^m$$

such that $u \in \mathcal{O}_{\mathbb{A}^1,0}^\times = \{f/g : f, g \in k[x],\ f(0), g(0) \neq 0\}$. So $\operatorname{ord}_0 \varphi = m$.

Remark. We can also think of $\mathcal{O}_{X,E} = \{\varphi \in K(X) : \operatorname{ord}_E(\varphi) \geq 0\}$.

Definition 11.1. For $0 \neq \varphi \in K(X)$, its *divisor of zeros and poles* is

$$\operatorname{div}(\varphi) = \operatorname{div}_X(\varphi) := \sum_{\substack{E \subseteq X \\ \text{prime}}} \operatorname{ord}_E(\varphi) E.$$

Proposition 11.2. *The divisor of zeros and poles* $\operatorname{div}(\varphi)$ *is a divisor, i.e. we have* $\operatorname{ord}_E(\varphi) = 0$ *for all but finitely many prime divisors* $E \subseteq X$.

Proof. Fix an open affine $\varnothing \neq U \subseteq X$. Since $X \setminus U$ contains at most finitely many prime divisors, it suffices to show that $\operatorname{div}_U(\varphi)$ is a divisor. Write $\varphi = f/g$ with $f, g \in \mathcal{O}_X(U) = A$. Since

$$\operatorname{div}_U(\varphi) = \operatorname{div}_U(f) - \operatorname{div}_U(g),$$

it suffices to consider the case when $\varphi \in \mathcal{O}_X(U)$. Now for $E \subseteq X$ prime with $E \cap U \neq \varnothing$, we have

$$\begin{aligned} \operatorname{ord}_E(\varphi) > 0 &\iff \varphi \in \mathfrak{m}_E \subseteq \mathcal{O}_{X,E} \\ &\iff \varphi \text{ vanishes on } E \cap U \\ &\iff E \subseteq V(\varphi). \end{aligned}$$

As $V(\varphi)$ contains finitely many prime divisors, we get that $\operatorname{div}_U(\varphi)$ is a divisor. $\square$

Example 11.1.1. Let $f \in k[x_1, \ldots, x_n] \in \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n)$. We can write

$$f = cf_1^{a_1} \cdots f_r^{a_r}$$

with f_i irreducible and $a_i \geq 0$. Then we have

$$\operatorname{div}_{\mathbb{A}^n}(f) = \sum_{i=1}^{r} a_i V(f_i).$$

This follows since $\operatorname{ord}_{V(f_i)} f_i = 1$, which in turns follows from $\mathcal{O}_{\mathbb{A}^n, V(f_i)} \cong k[x_1, \ldots, x_n]_{(f_i)}$, which has maximal ideal (f_i) and hence uniformizer f_i.

Proposition 11.3 (Algebraic Hartog's lemma)**.** *For $0 \neq \varphi \in K(X)$, we have $\operatorname{div} \varphi \geq 0$ (i.e. $\operatorname{ord}_E \varphi \geq 0$ for all prime divisors $E \subseteq X$) if and only if $\varphi \in \mathcal{O}_X(X)$.*

Proof. It suffices to show the statement on an affine cover. So we may assume that X is affine. Write $A = \mathcal{O}_X(X)$. Hartog's lemma in commutative algebra implies that

$$A = \bigcap_{\substack{\mathfrak{p} \leq A \text{ prime} \\ \text{height } 1}} A_{\mathfrak{p}} \subseteq \operatorname{Frac}(A).$$

Thus for $\varphi \in \operatorname{Frac}(A) \cong K(X)$, we have $\varphi \in A$ if and only if $\varphi \in A_{\mathfrak{p}}$ for all height 1 prime ideals $\mathfrak{p} \leq A$, which happens if and only if $\operatorname{ord}_E(\varphi) \geq 0$ for all prime divisors $E \subseteq X$, i.e. $\operatorname{div}_X(\varphi) \geq 0$. $\square$

11.2 Class Groups

Definition 11.2. A divisor $D \in \operatorname{Div} X$ is *principal* if $D = \operatorname{div}_X(\varphi)$ for some $0 \neq \varphi \in K(X)$.

Example 11.2.1. Any divisor on $\mathbb{A}^n$ is principal by the previous examples (this is also true when $\mathbb{A}^n$ is replaced by an affine variety X such that $\mathcal{O}_X(X)$ is a UFD).

Remark. Note that $\operatorname{PDiv} X = \{\operatorname{div} \varphi : 0 \neq \varphi \in K(X)\} \subseteq \operatorname{Div} X$ is a subgroup.

Definition 11.3. We say that $D_1, D_2 \in \operatorname{Div} X$ are *linearly equivalent* if $D_1 - D_2$ is principal. The *class group* of X is

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{PDiv}(X) = \text{group of divisors up to } \sim.$$

Example 11.3.1. We have the following:

1. $\operatorname{Cl}(\mathbb{A}^n) = 0$.
2. $\operatorname{Cl}(\mathbb{P}^n) = \mathbb{Z}[H]$, where $H \subseteq \mathbb{P}^n$ is a hyperplane.

   To see this, recall that we have a bijection

   $$\{\text{prime divisors in } \mathbb{P}^n\} \longleftrightarrow \{\text{irreducible hypersurfaces}\}.$$

   Consider $\deg : \operatorname{Div}(\mathbb{P}^n) \to \mathbb{Z}$ which sends $\sum a_i E_i = \sum a_i \deg E_i$, which is clearly surjective. For $\varphi \in K(\mathbb{P}^n)$, write $\varphi = f/g$ where $f, g \in k[x_0, \ldots, x_n]$ are homogeneous of the same degree. So

   $$f = f_1^{a_1} \cdots f_r^{a_r} \quad \text{and} \quad g = g_1^{b_1} \cdots g_s^{b_s}$$

with f_i, g_i homogeneous and $a_i, b_i \in \mathbb{Z}_{\geq 0}$, and $\sum a_i \deg f_i = \sum b_i \deg g_i$. Now

$$\operatorname{div} \varphi = \sum_{i=1}^{r} a_i V(f_i) - \sum_{i=1}^{s} b_i V(g_i).$$

So $\deg \varphi = 0$. This shows that $\operatorname{PDiv}(\mathbb{P}^n) \subseteq \ker \deg$. One can show the reverse inclusion as well, so $\operatorname{PDiv}(\mathbb{P}^n) = \ker \deg$. Thus we have $\operatorname{Cl}(\mathbb{P}^n) = \operatorname{Div}(\mathbb{P}^n)/\operatorname{PDiv}(\mathbb{P}^n) \cong \mathbb{Z}$.

11.3 Cartier Divisors

Definition 11.4. A divisor D on X is *Cartier* if it is locally principal, i.e. at each $x \in X$, there exists an open neighborhood $x \in U_x \subseteq X$ and $0 \neq \varphi_x \in K(X)$ such that $\operatorname{div}_{U_x}(\varphi_x) = D|_{U_x}$.

Example 11.4.1. Principal divisors are Cartier (in fact, they are globally principal).

Example 11.4.2. Let $H = V(x_0) \subseteq \mathbb{P}^n$. Note $H \nsim 0$, so H is not principal. But on $U_i = \{x_i \neq 0\} \subseteq \mathbb{P}^n$,

$$H|_{U_i} = \operatorname{div}_{U_i}(x_0/x_i),$$

so H is Cartier (alternatively, $\mathbb{P}^n$ has an open cover by the U_i, and each $\operatorname{Cl}(U_i) = 0$).

Proposition 11.4. *Every divisor D on X is Cartier.*

Proof. If D_1, D_2 are Cartier, then so is $D_1 + D_2$. So it suffices to consider the case when $D = E$ for some prime divisor E on X. Now fix $x \in X$. If $x \notin E$, then set $U_x = X \setminus E$ and $\varphi_x = 1$. Now assume $x \in E$. Since $\mathcal{O}_{X,x}$ is a regular local ring, the Auslander-Buchsbaum theorem implies that $\mathcal{O}_{X,x}$ is a UFD. Now we have a bijection

$$\{\text{prime ideals of } \mathcal{O}_{X,x}\} \longleftrightarrow \{\text{irreducible closed subvarieties } x \in Y \subseteq X\}.$$

Write $J \subseteq \mathcal{O}_{X,x}$ for the prime ideal corresponding to E. As J is a height 1 prime ideal and $\mathcal{O}_{X,x}$ is a UFD, there exists $f \in \mathcal{O}_{X,x}$ such that $J = (f)$. So there exists an isomorphism

$$\varphi : (\mathcal{O}_{X,x})_{(f)} \xrightarrow{\cong} \mathcal{O}_{X,E}$$

such that $\varphi(f)\mathcal{O}_{X,E} = \mathfrak{m}_E$. So $\operatorname{ord}_E(f) = 1$. One then checks that $\operatorname{div}_{U_x}(f) = E|_{U_x}$ for some U_x. $\square$

Remark. Proposition 11.4 crucially uses our assumption that X is smooth. It may fail in general.

Lecture 12

Feb. 19 — Divisors, Part 2

12.1 Cartier Divisors, Continued

Example 12.0.1. Consider $X = V(xy - z^2)$. Note that this is singular. Also, all of the local rings for this particular variety are still DVRs, so we can still talk about divisors. The prime divisors are

$$D_1 = V(x, z) \quad \text{and} \quad D_2 = V(y, z).$$

One can compute that $\operatorname{div}_X(x) = 2D_1$, $\operatorname{div}_X(y) = 2D_2$, and $\operatorname{div}_X(z) = D_1 + D_2$. So D_1 is not Cartier.

Example 12.0.2. Consider $X = V(xy - zw)$, which is also singular. Then $D = V(x, z)$ is a prime divisor, and one can check that mD is not Cartier for all $m \neq \mathbb{Z} \setminus \{0\}$.

12.2 Divisors and Invertible Sheaves

Definition 12.1. For $D \in \operatorname{Div} X$, define $\mathcal{O}_X(D) \in \operatorname{Mod}_{\mathcal{O}_X}$ such that for $U \subseteq X$ open,

$$\mathcal{O}_X(D)(U) = \{\varphi \in K(X) : \operatorname{div}_U(\varphi) + D|_U \geq 0\}$$

Remark. If $D \geq 0$, then one can think of this as saying that "the poles of φ are $\leq D$."

Remark. Note that $\mathcal{O}_X(D) \subseteq \underline{K(X)}$ is a subsheaf.

Example 12.1.1. Let $H = V(x_1) \subseteq \mathbb{A}^n$. Then

$$\begin{aligned}\mathcal{O}_{\mathbb{A}^n}(H)(\mathbb{A}^n) &= \{\varphi \in k[x_1, \ldots, x_n] : \operatorname{ord}_H(\varphi) \geq -1 \text{ and } \operatorname{ord}_E(\varphi) \geq 0 \text{ for all prime } H \neq E \subseteq X\} \\ &= \frac{1}{x_1} k[x_1, \ldots, x_n] \subseteq k(x_1, \ldots, x_n).\end{aligned}$$

Proposition 12.1. *If D is principal and $D = \operatorname{div}_X(g)$, then*

$$\mathcal{O}_X(D) = \frac{1}{g}\mathcal{O}_X \subseteq \underline{K(X)}.$$

Proof. Fix $U \subseteq X$ open. For $\varphi \in K(X)$, we have $\varphi \in \mathcal{O}_X(D)(U)$ if and only if

$$\operatorname{div}_U(\varphi) + D|_U = \operatorname{div}_U(\varphi) + \operatorname{div}_U(g) \geq 0,$$

if and only if $\operatorname{div}_U(\varphi g) \geq 0$, if and only if $\varphi g \in \mathcal{O}_X(U)$, if and only if $\varphi = \psi/g$ for some $\varphi \in \mathcal{O}_X(U)$. So we have $\mathcal{O}_X(D)(U) = (1/g)\mathcal{O}_X(U)$. $\square$

Corollary 12.0.1. *For any $D \in \operatorname{Div} X$, $\mathcal{O}_X(D)$ is an invertible $\mathcal{O}_X$-module.*

Proof. Since D is necessarily Cartier (recall the smoothness assumption), there exists an open cover $X = U_1 \cup \cdots \cup U_r$ and $g_i \in K(X)^\times$ such that $D|_{U_i} = \operatorname{div}_{U_i}(g_i)$. Now

$$\mathcal{O}_X(D)|_{U_i} = \mathcal{O}_{U_i}(D|_{U_i}) = \frac{1}{g_i}\mathcal{O}_{U_i}.$$

Now we have an isomorphism

$$\begin{aligned} \alpha : \mathcal{O}_X(D)|_{U_i} = \frac{1}{g_i}\mathcal{O}_{U_i} &\longrightarrow \mathcal{O}_{U_i} \\ f &\longmapsto fg_i. \end{aligned}$$

So $\mathcal{O}_X(D)$ is an invertible $\mathcal{O}_X$-module with trivialization data $(U_i, g_{i,j} = g_i/g_j)$. □

Example 12.1.2. If $H = V(x_0) \subseteq \mathbb{P}^n$, then we have

$$H|_{U_i} = \operatorname{div}_{U_i}(x_0/x_i),$$

so $\mathcal{O}_{\mathbb{P}^n}(H)$ is the invertible sheaf with transition data $(U_i, x_j/x_i)$. So $\mathcal{O}_{\mathbb{P}^n}(H) \cong \mathcal{O}_{\mathbb{P}^n}(1)$.

Proposition 12.2. *The map*

$$\begin{aligned} \operatorname{Div} X &\longrightarrow \operatorname{Pic} X \\ D &\longmapsto \mathcal{O}_X(D) \end{aligned}$$

is a surjective homomorphism with kernel $\operatorname{PDiv} X$. *In particular,* $\operatorname{Cl}(X) \cong \operatorname{Pic} X$, *and* $D_1 \sim D_2$ *if and only if* $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.

Proof. We first check that this map is a group homomorphism. Fix $D_1, D_2 \in \operatorname{Div} X$. Choose an open cover $\{U_i\}$ of X such that there exist $g_i^1, g_i^2 \in K(X)^\times$ with

$$D_1|_{U_i} = \operatorname{div}_{U_i}(g_i^1) \quad \text{and} \quad D_2|_{U_i} = \operatorname{div}_{U_i}(g_i^2).$$

So we have $(D_1 + D_2)|_{U_i} = \operatorname{div}_{U_i}(g_i^1 g_i^2)$. Then we have:

invertible sheaf	transition data
$\mathcal{O}_X(D_1)$	g_i^1/g_j^1
$\mathcal{O}_X(D_2)$	g_i^2/g_j^2
$\mathcal{O}_X(D_1 + D_2)$	$g_i^1 g_i^2/g_j^1 g_j^2$

So we see that $\mathcal{O}_X(D_1 + D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$.

Now we check the kernel. If $D \sim 0$, then $D = \operatorname{div}_X g$ for some $g \in K(X)^\times$. So

$$\mathcal{O}_X(D) = \frac{1}{g}\mathcal{O}_X \cong \mathcal{O}_X.$$

Conversely, assume that there exists an isomorphism $\beta : \mathcal{O}_X \to \mathcal{O}_X(D)$. Set $f = \beta(1)$. Since $1\mathcal{O}_X = \mathcal{O}_X$,

$$f\mathcal{O}_X = \mathcal{O}_X(D).$$

So $f\mathcal{O}_{U_i} = (1/g_i)\mathcal{O}_{U_i}$, so $fg_i \in \mathcal{O}_X(U_i)^\times$. Then

$$\operatorname{div}_{U_i} f = \operatorname{div}_{U_i}(1/g_i) = -D|_{U_i}.$$

So $\operatorname{div}_X f = -D$, and thus D is principal.

Finally, we check surjectivity. Fix an invertible sheaf $\mathcal{L} \in \operatorname{Pic} X$, say with trivialization data $(U_i, g_{i,j})_{i\in I}$. Fix an element $0 \in I$ and set $g_0 = 1$, $g_i = g_{i,0}$ for $i \in I \setminus \{0\}$. Then

$$g_i/g_j = g_{i,0}/g_{j,0} = g_{i,j} \in \mathcal{O}_X(U_{i,j})^\times.$$

Let $D^i = \operatorname{div}_{U_i}(g_i) \in \operatorname{Div}(U_i)$. Note that

$$D^i_{U_{i,j}} - D^j_{U_{i,j}} = \operatorname{div}_{U_{i,j}}(g_i/g_j) = \operatorname{div}_{U_{i,j}}(g_{i,j}) = 0,$$

so the D^i glue to a divisor $D \in \operatorname{Div} X$ such that $D|_{U_i} = \operatorname{div}_{U_i}(g_i)$. So $\mathcal{O}_X(D) \cong \mathcal{L}$. □

Remark. Proposition 12.2 implies the following:

- $\operatorname{Pic}(\mathbb{A}^n) \cong \operatorname{Cl}(\mathbb{A}^n) = 0$.
- $\operatorname{Pic}(\mathbb{P}^n) \cong \operatorname{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$, and $\operatorname{Pic}(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}(1)\rangle$.

12.3 Effective Cartier Divisors

Remark. Fix $D = \sum a_i D_i \in \operatorname{Div} X$ such that the D_i are distinct prime divisors.

Definition 12.2. A divisor D is called *effective* if $a_i \geq 0$ for all i, and we write $D \geq 0$.

Remark. If $D \geq 0$, then we have

$$\mathcal{O}_X(-D) \subseteq \mathcal{O}_X \subseteq \underline{K(X)},$$

and $\mathcal{O}_X(-D)$ is a coherent ideal sheaf. We write $\mathcal{O}_D := \mathcal{O}_X/\mathcal{O}_X(-D)$.

Example 12.2.1. Let $H = V(x_1) \subseteq \mathbb{A}^n$. Then $\mathcal{O}_{\mathbb{A}^n}(-H) = x_1\mathcal{O}_{\mathbb{A}^n}$, and

$$\mathcal{O}_H = (k[x_1, \ldots, x_n]/(x_1))^\sim = i_*\mathcal{O}_H,$$

where $i : H \hookrightarrow \mathbb{A}^n$ is the inclusion. In general, we have

1. $\operatorname{Supp}(\mathcal{O}_D) = \bigcup_{i=1}^r E_i$, where $D = \sum_{i=1}^r b_i E_i$ with $b_i > 0$ and E_i distinct primes.

   To see this, we can write $(\mathcal{O}_D)_x = \mathcal{O}_{X,x}/\mathcal{O}_X(-D)_x$, which is zero if and only if $x \notin \bigcup_{i=1}^r E_i$. We will call this locus the *support* of D.

2. If $b_1 = \cdots = b_r = 1$, then

   $$\mathcal{O}_X(-D) = \mathcal{I}_{E_1\cup\cdots\cup E_r}.$$

   Thus $\mathcal{O}_D = i_*\mathcal{O}_{E_1\cup\cdots\cup E_r}$, where, where $i : E_1 \cup \cdots \cup E_r \hookrightarrow X$ is the inclusion.

12.4 Effective Divisors and Zero Loci of Sections

Remark. Fix an invertible sheaf $\mathcal{L}$ on X. Fix trivialization data

$$(U_i, \alpha_i : \mathcal{L}|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}).$$

Proposition 12.3. *For $0 \neq s \in \Gamma(X, \mathcal{L})$, there exists a divisor $D \in \operatorname{Div} X$ such that*

1. $D|_{U_i} = \operatorname{div}_{U_i}(\alpha_i(s))$,
2. $D \geq 0$,
3. $\mathcal{O}_X(D) \cong \mathcal{L}$.

Write $V(s) := D$ for this divisor.

Proof. (1) It suffices to show $\operatorname{div}_{U_{i,j}}(\alpha_i(s)) = \operatorname{div}_{U_{i,j}}(\alpha_j(s))$, and this holds as $\alpha_i(s)/\alpha_j(s) \in \mathcal{O}_X(U_{i,j})^\times$.
(2) Use that $\alpha_i(s) \in \mathcal{O}_X(U_i)$.
(3) Left as an exercise. One works with transition functions. □

Remark. We get a bijection

$$\begin{aligned}\{\text{1-dimensional subspaces in } \Gamma(X,\mathcal{L})\} &\longleftrightarrow \{\text{effective Cartier divisors } D \text{ such that } \mathcal{O}_X(D) \cong \mathcal{L}\}\\ ks &\longmapsto V(s)\\ \mathcal{L} \cong \mathcal{O}_X(D) \ni 1 &\longleftarrow\!\shortmid D.\end{aligned}$$

If $\mathcal{L} \cong \mathcal{O}_X(D_0)$, then there is another bijection with the effective Cartier divisors $D \sim D_0$.

Example 12.2.2. Let $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}_X(m)$, $m \geq 1$. Then $\mathcal{L} \cong \mathcal{O}_X(mH)$ for a hyperplane H, and

$$\begin{aligned}\{\text{effective Cartier divisors } D \sim mH\} &= \left\{ \sum a_i H_i : \begin{array}{c} a_i \geq 0,\ H_i \text{ irreducible hypersurfaces} \\ \text{with } \sum a_i \deg H_i = m \end{array} \right\} \\ &= \{\text{hypersurfaces of degree } m \text{ in } \mathbb{P}^n \text{ counting multiplicities}\}.\end{aligned}$$

Remark. If $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ globally generate $\mathcal{L}$,

$$\begin{aligned} f : X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \cdots : s_n(x)], \end{aligned}$$

and $H_a = V(\sum a_i x_i) \subseteq \mathbb{P}^n$ is a hyperplane, then $f^{-1}(H_a) = V(\sum a_i s_i)$.

Lecture 13

Feb. 24 — Differentials

13.1 Review of Tangent Spaces

Definition 13.1. For a variety X and $x \in X$, the *tangent space* of X at x is

$$T_{X,x} = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee.$$

We say that X is *smooth* at x if $\dim_x X = \dim_k T_{X,x}$.

Remark. Our goals will be:

1. Define $\Omega_X \in \mathrm{Coh}(X)$ such that $\Omega_X(U)$ gives the "algebraic 1-forms on U."
2. Define $T_X := \Omega_X^\vee$.
3. Show that there is a natural isomorphism $T_X(x) \cong T_{X,x}$.
4. Construct a relative version $\Omega_{X/Y}$ for a morphism $X \to Y$.

13.2 Differentials

Remark. Let $\phi : A \to B$ be a morphism of rings, e.g. $k \to k[x_1, \ldots, x_n]/I$.

Definition 13.2. For a B-module M, an A-*derivation* is a map $D : B \to M$ such that

1. D is A-linear,
2. (Leibniz rule) $D(b_1 b_2) = b_1 D(b_2) + D(b_1) b_2$.

Remark. Using (2), condition (1) can be reformulated as the following:

1'. D is additive and $D(a \cdot 1) = 0$ for all $a \in A$.

Example 13.2.1. Let $A = k$ and $B = k[x_1, \ldots, x_n]$. We can define

$$\begin{aligned} D : k[x_1, \ldots, x_n] &\longrightarrow \bigoplus_{i=1}^n k[x_1, \ldots, x_n] dx_i \\ f &\longmapsto df = \sum \frac{\partial f}{\partial x_i} dx_i. \end{aligned}$$

One can check that D is a k-derivation (but D is not $k[x_1, \ldots, x_n]$-linear).

Proposition 13.1. *There exists a derivation $B \to \Omega_{B/A}$ such that for any derivation $D : B \to M$, there exists unique a B-module homomorphism such that the following diagram commutes:*

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "D"'] & \Omega_{B/A} \arrow[d, dashed, "\exists!"] \\ & M \end{tikzcd}$$

Here $\Omega_{B/A}$ is called the module of Kähler differentials.

Proof. Let $\Omega_{B/A}$ be the B-module with generators $(db)_{b\in B}$ and relations

1. $d(b_1b_2) = b_1d(b_2) + b_2d(b_1)$ for $b_1, b_2 \in B$,
2. $d(b_1 + b_2) = db_1 + db_2$ for $b_1, b_2 \in B$ and $d(\phi(a)) = 0$ for $a \in A$.

Then one can check $B \to \Omega_{B/A}$ given by $b \mapsto db$ is a derivation satisfying the universal property. □

Remark. If $(b_i)_{i\in I}$ generate B as an A-module, then the db_i generate $\Omega_{B/A}$ as a B-module.

Example 13.2.2. We have the following:

0. $\Omega_{A/A} = 0$.
1. For $K \subseteq L$ a finite field extension, $\Omega_{L/K} = 0$ if and only if L/K is separable.
2. Let $B = A[x_1, \ldots, x_n]$. We know that dx_i for $1 \le i \le n$ generate $\Omega_{B/A}$. We claim that the dx_i are also linearly independent over B.

   To see this, fix $1 \le i \le n$ and consider the derivation $D_i : B \to B$ given by $f \mapsto \partial f/\partial x_i$. Then

$$D_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

   By the universal property we have a diagram

$$\begin{tikzcd} & \Omega_{B/A} \arrow[d, dashed] \\ B \arrow[ur] \arrow[r, "D_i"'] & B \end{tikzcd}$$

   so the dx_i are independent over B. Thus we have $\Omega_{B/A} = Bdx_1 \oplus \cdots \oplus Bdx_n$.

Proposition 13.2. *If $S \subseteq B$ is a multiplicative system, then there is a canonical isomorphism*

$$\begin{aligned} \Omega_{S^{-1}B/A} &\xrightarrow{\cong} S^{-1}\Omega_{B/A} \\ d(b/s) &\longmapsto \frac{1}{s}db - \frac{b}{s^2}ds. \end{aligned}$$

Proof. Given an A-derivation $S^{-1}B \to M$, we get an A-derivation by composition:

$$B \longrightarrow S^{-1}B \longrightarrow M$$

So there exists a unique B-module homomorphism D such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, "d"] \arrow[d] & \Omega_{B/A} \arrow[d, dashed, "D"] \arrow[dr] & \\
S^{-1}B \arrow[r] \arrow[rr, bend right, "d'"'] & M & S^{-1}\Omega_{B/A} \arrow[l, dashed, "S^{-1}D"]
\end{tikzcd}$$

Note that by the universal property of localization, we get a unique map $S^{-1}D$ that fits into the above diagram. Then one can check that the map

$$\begin{aligned} d' : S^{-1}B &\longrightarrow S^{-1}\Omega_{B/A} \\ \frac{b}{s} &\longmapsto \frac{1}{s}db - \frac{b}{s^2}ds \end{aligned}$$

makes the diagram commute and is an A-derivation, so $S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A}$. □

Proposition 13.3. *We have the following:*

1. *For A-algebras B_1 and B_2, we have $\Omega_{(B_1\otimes_A B_2)/A} \cong (\Omega_{B_1/A} \otimes_A B_2) \oplus (B_1 \otimes_A \Omega_{B_2/A})$.*
2. *If we have ring homomorphisms $A \xrightarrow{\phi} B \xrightarrow{\psi} C$, then the sequence*

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

is exact, where the first map $\Omega_{B/A} \otimes_B C \to \Omega_{C/A}$ is given by $db \otimes c \mapsto cd\psi(b)$.

3. *If $I \leq B$ is an ideal, then there exists an exact sequence*

$$I/I^2 \longrightarrow \Omega_{B/A} \otimes (B/I) \longrightarrow \Omega_{(B/I)/A} \longrightarrow 0$$

where the first map $I/I^2 \to \Omega_{B/A} \otimes (B/I)$ is given by $\bar{b} \mapsto db \otimes 1$.

Proof. See Mustata's notes. □

Example 13.2.3. Let $X \subseteq \mathbb{A}^n$ be an affine variety, $C = \mathcal{O}_X(X) = k[x_1, \dots, x_n]/I(X)$, $B = k[x_1, \dots, x_n]$, and $A = k$. Let $I = I(X)$. Then Proposition 13.3(3) gives an exact sequence

$$I/I^2 \longrightarrow \bigoplus_{i=1}^n Bdx_i \otimes_B C \longrightarrow \Omega_{C/k} \longrightarrow 0$$

Thus we see that $\Omega_{C/k}$ is given by

$$\Omega_{C/k} = \frac{\bigoplus_{i=1}^n Cdx_i}{(df : f \in I)}.$$

If $I = (f_1, \dots, f_r)$, then we just get

$$\Omega_{C/k} = \frac{\bigoplus_{i=1}^n Cdx_i}{(df_1, \dots, df_r)}.$$

Example 13.2.4. If $X = V(x^2 - y^3)$, then $C = \mathcal{O}_X(X) = k[x, y]/(x^2 - y^3)$, and Example 13.2.3 gives

$$\Omega_{C/k} = \frac{Cdx \oplus Cdy}{(2xdx - 3y^2dy)}.$$

Proposition 13.4. *Given a variety X, there exists a coherent sheaf Ω_X, called the* sheaf of differentials, *such that*

1. *for $U \subseteq X$ open affine, there exists an isomorphism $\Omega_X(U) \cong \Omega_{\mathcal{O}_X(U)/k}$,*
2. *for $V \subseteq U \subseteq X$ open affine, the following diagram commutes:*

$$\begin{tikzcd} \Omega_X(U) \arrow[r] \arrow[d] & \Omega_{\mathcal{O}_X(U)/k} \arrow[d, "df \mapsto d(f|_U)"] \\ \Omega_X(V) \arrow[r] & \Omega_{\mathcal{O}_X(V)/k} \end{tikzcd}$$

Proof. For $U \subseteq X$ open affine, set $\mathcal{G}_U := \widetilde{\Omega_{\mathcal{O}_X(U)/k}} \in \mathrm{Coh}(U)$. Now for $V \subseteq U \subseteq X$ open affine,

$$\Omega_{\mathcal{O}_X(U)/k} \longrightarrow \Omega_{\mathcal{O}_X(V)/k}$$

induces a morphism $\phi_{V,U} : \mathcal{G}_U|_V \to \mathcal{G}_V$. If $V = D_U(f)$, then

$$\mathcal{G}_U|_V = ((\Omega_{\mathcal{O}_X(U)/k})_f)^\sim \quad \text{and} \quad \mathcal{G}_V = \widetilde{\Omega_{\mathcal{O}_X(V)/k}}.$$

These are isomorphic by the localization property of differentials. So by the affine communication lemma, we get that $\phi_{V,U}$ is an isomorphism for all $V \subseteq U \subseteq X$ open affine.

Using this, we can glue the $\mathcal{G}_U$ for $U \subseteq X$ open affine to construct $\mathcal{G} =: \Omega_X$. $\square$

Remark. Similarly, for a morphism of varieties $f : X \to Y$, we can define

$$\Omega_{X/Y} \in \mathrm{Coh}(X)$$

such that for any open affines $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, we have

$$\Omega_{X/Y}|_U \cong (\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)})^\sim.$$

Remark. Note that $\Omega_X \cong \Omega_{X/\{\mathrm{pt}\}}$.

Example 13.2.5. We have the following:

1. $\Omega_{\mathbb{A}^n} \cong (\Omega_{k[x_1,\dots,x_n]/k})^\sim \cong (\bigoplus_{i=1}^n k[x_1,\dots,x_n]dx_i)^\sim \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{A}^n}$.
2. To compute $\Omega_{\mathbb{P}^n}$, note that there is a short exact sequence (called the *Euler sequence*)

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

On $U_i \subseteq \mathbb{P}^n$, this corresponds to

$$\begin{array}{ccccccc} \Omega_{\mathbb{P}^n}(U_i) & \longrightarrow & \bigoplus_{j=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(-1)(U_i) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(U_i) & \longrightarrow & 0 \\ d(z_j/z_i) & \longmapsto & e_j - (z_j/z_i)e_i & & & & \\ & & e_j & \longmapsto & z_j/z_i. & & \end{array}$$

One can check that these maps glue and that the sequence is exact.

Lecture 14

Feb. 26 — Differentials, Part 2

14.1 Tangent Sheaf

Proposition 14.1. *We have the following:*

1. *For $X \subseteq \mathbb{A}^n$ closed, we have*
$$\Omega_X(X) = \frac{\bigoplus_{i=1}^n A dx_i}{(df_1, \ldots, df_r)},$$
where $A = \mathcal{O}_X(X) = k[x_1, \ldots, x_n]/I(X)$ and $I(X) = (f_1, \ldots, f_r) \leq k[x_1, \ldots, x_n]$.
2. *For $x \in X$, we have $(\Omega_X)_x = \Omega_{\mathcal{O}_{X,x}/k}$.*

Proof. (1) This was the discussion in Example 13.2.3.

(2) We may assume that X is affine. Let $I := I_X(\{x\}) \leq \mathcal{O}_X(X)$. Then
$$(\Omega_X)_x \cong (\widetilde{\Omega_{\mathcal{O}_X(X)/k}})_x \cong (\Omega_{\mathcal{O}_X(X)/k})_I = \Omega_{\mathcal{O}_{X,x}/k},$$
which is the desired claim. $\square$

Proposition 14.2. *For a variety X and $x \in X$,*

1. *$T_{X,x}^\vee \cong \Omega_X(x)$, in particular $\dim \Omega_X(x) = \dim_k T_{X,x}$;*
2. *X is smooth if and only if Ω_X is locally free (of rank $\dim X$ if X is irreducible).*

In particular, when (2) holds, $T_X = \Omega_X^\vee$ is a locally free sheaf.

Proof. (1) We will only show the dimension statement. As both are local statements, we may assume that $x = 0 \in X \subseteq \mathbb{A}^n$ where X is closed. Write $A = \mathcal{O}_X(X)$ and $I_X = (f_1, \ldots, f_r) \leq k[x_1, \ldots, x_n]$. Then
$$\Omega_X(0) = \frac{\Omega_X(X)}{(\overline{x}_1, \ldots, \overline{x}_n)\Omega_X(X)} = \frac{\bigoplus_{i=1}^n k dx_i}{(\sum_j (\partial f_i/\partial x_j)(0) dx_j)}.$$
by Proposition 14.1. As $T_{X,0} = V(f_1^{\text{lin}}, \ldots, f_r^{\text{lin}})$, we see that $\dim \Omega_X(0) = \dim T_{X,0}$.

(2) We may assume that X is connected. If X is smooth, then X is irreducible, so
$$n := \dim X = \operatorname{codim}_X\{x\} = \dim T_{X,x} = \dim \Omega_X(x)$$
for all $x \in X$. Thus a homework problem implies that Ω_X is free.

Conversely, assume Ω_X is free. Set $n = \operatorname{rk} \Omega_X$, then for $x \in X^{\mathrm{sm}}$, we have $\operatorname{codim}_X\{x\} = \dim \Omega_X(x) = n$. Now X^{sm} is dense in X, so X is of pure dimension n. So for all $x \in X$, we have

$$n = \dim X = \operatorname{codim}_X\{x\} \leq \dim T_{x,x} = \dim \Omega_X(x) = n,$$

hence $\dim T_{X,x} = n$, so X is smooth. $\square$

Definition 14.1. For a variety X, its *tangent sheaf* (or *tangent bundle*) is $T_X := \Omega_X^\vee$.

Remark. If X is smooth, then T_X is locally free.

Conjecture 14.0.1 (Zariski-Lipman)**.** *If T_X is locally free, then X is smooth.*

Remark. In general, $\mathcal{F}$ being locally free does not imply that $\mathcal{F}^\vee$ is locally free.

Lecture 15

Mar. 3 — Differential Forms

15.1 Differential Forms

Definition 15.1. For a variety X and $0 \leq p \leq \dim X$,

$$\Omega_X^p := \wedge^p \Omega_X$$

is the space of *p-forms* on X. If X is smooth and irreducible, its *canonical bundle* is

$$\omega_X := \wedge^{\dim X} \Omega_X = \det(\Omega_X).$$

Remark. If M is a smooth compact oriented manifold, the *de Rham cohomology* is given by

$$H_{\mathrm{dR}}^k(M, \mathbb{R}) = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.$$

Then Poincaré duality says that (here $n = \dim X$)

$$\begin{aligned} H_{\mathrm{dR}}^k(M, \mathbb{R}) \times H_{\mathrm{dR}}^{n-k}(M, \mathbb{R}) &\longrightarrow H_{\mathrm{dR}}^n(M, \mathbb{R}) \cong \mathbb{R} \\ ([\alpha], [\beta]) &\longmapsto [\alpha \wedge \beta] \end{aligned}$$

is a perfect pairing, i.e. the following map is an isomorphism:

$$\begin{aligned} H_{\mathrm{dR}}^k(M, \mathbb{R}) &\longrightarrow H_{\mathrm{dR}}^{n-k}(M, \mathbb{R})^\vee \\ [\alpha] &\longmapsto ([\beta] \mapsto [\alpha \wedge \beta]) \end{aligned}$$

The canonical bundle ω_X will be an analogue of $H_{\mathrm{dR}}^n(M, \mathbb{R}) \cong \mathbb{R}$ in algebraic geometry.

15.2 Canonical Divisor

Definition 15.2. The *canonical divisor* of a smooth irreducible variety X is a divisor K_X such that $\omega_X \cong \mathcal{O}_X(K_X)$ (note that technically K_X is only defined up to linear equivalence).

Example 15.2.1. How do we compute the canonical divisor K_X?

1. First consider $\mathbb{A}^n$. In this case $\Omega_{\mathbb{A}^n} = \mathcal{O}_{\mathbb{A}^n}^{\oplus n}$, so $\det(\Omega_{\mathbb{A}^n}) = \mathcal{O}_{\mathbb{A}^n}$. So $K_{\mathbb{A}^n} = 0$ (alternatively one can note that $\mathrm{Cl}(\mathbb{A}^n) = 0$, so there every divisor is 0 up to linear equivalence).
2. Next consider $\mathbb{P}^n$. We need to use the following tools:
   i) The Euler short exact sequence $0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$.
   ii) If $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ is a short exact sequence of locally free sheaves of ranks e, f, g on X, then $\det(\mathcal{F}) \cong \det(\mathcal{E}) \otimes \det(\mathcal{G})$.

To see this, note that given a short exact sequence of free R-modules

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

of ranks e, f, g for some ring R, we can define a map

$$\begin{aligned} \det(E) \otimes \det(G) &\longrightarrow \det(F) \\ (v_1 \wedge \cdots \wedge v_e) \otimes (\overline{w}_1 \wedge \cdots \wedge \overline{w}_g) &\longmapsto v_1 \wedge \cdots \wedge v_e \wedge w_1 \wedge \cdots \wedge w_g, \end{aligned}$$

where we think of $G = F/E$. One can check that this is well-defined, is an isomorphism (it is enough to check this on a basis), and commutes with localization. This induces a morphism

$$\det(\mathcal{F})|_U \longrightarrow \det(\mathcal{E})|_U \otimes \det(\mathcal{G})|_U$$

on $U \subseteq X$ open affine which trivializes $\mathcal{E}, \mathcal{F}, \mathcal{G}$. Since the map commutes with localization, these morphisms glue to give an isomorphism $\det(\mathcal{F}) \to \det(\mathcal{E}) \otimes \det(\mathcal{G})$.

iii) For $\mathcal{L}_1, \ldots, \mathcal{L}_r$ invertible sheaves, we have

$$\det(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r) \cong \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_r.$$

One can prove this as above by defining a map on affine opens and showing that it commutes with localization, or by checking the transition data.

Using these three properties, we see that

$$\omega_{\mathbb{P}^n} = \det \Omega_{\mathbb{P}^n} = \det(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}) \otimes \det(\mathcal{O}_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Hence we see that $K_{\mathbb{P}^n} = -(n+1)H$ with $H \subseteq \mathbb{P}^n$ a hyperplane.

15.3 Exact Sequences of Differentials

Proposition 15.1. *We have the following:*

1. *For a morphism of varieties $f : X \to Y$, there is an exact sequence*

$$f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

2. *For $i : Z \hookrightarrow X$ a closed embedding, there is an exact sequence (where $\mathcal{I} = \mathcal{I}_Z \subseteq \mathcal{O}_X$)*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_X \longrightarrow \Omega_Z \longrightarrow 0$$

Proof. (1) For $U \subseteq X$ open affine with $f(U) \subseteq V$ open affine, set

$$A = k \hookrightarrow B = \mathcal{O}_Y(V) \hookrightarrow C = \mathcal{O}_X(U).$$

From this, we get an exact sequence

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

Then use that this sequence is natural (i.e. for another row $A' \hookrightarrow B' \hookrightarrow C'$ with the appropriate arrows, there is another row in the exact sequence which commutes) and commutes with localization to get an exact sequence of coherent sheaves.

(2) Reduce to the corresponding ring theoretic statement. □

Remark. In Proposition 15.1(2), we abusively write $\mathcal{I}/\mathcal{I}^2$ for $i^*(\mathcal{I}/\mathcal{I}^2)$, since for X affine, $A = \mathcal{O}_X(X)$, and $I = I(Z) \leq A$, we have that I/I^2 is naturally an A/I-module as $\mathrm{Ann}(I/I^2) \subseteq I$. Thus

$$i^*(\mathcal{I}/\mathcal{I}^2) = (I/I^2 \otimes_A A/I)^{\sim} = \widetilde{I/I^2}.$$

More generally, there is an equivalence of categories

$$\mathrm{Coh}_Z \underset{i^*}{\overset{i_*}{\rightleftarrows}} \text{coherent sheaves } \mathcal{F} \text{ on } X \text{ with } \mathrm{Ann}(\mathcal{F}) \subseteq \mathcal{I}$$

Example 15.2.2. For $i : \{p\} \hookrightarrow \mathbb{P}^1$, we have an exact sequence

$$0 \longrightarrow \mathcal{I}_{\{p\}} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow i_*\mathcal{O}_{\{p\}} \longrightarrow 0$$

Remark. If Z, X are smooth varieties of pure dimension, then

- Mustata 6.3.21 implies that $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_X$-module of rank $\dim X - \dim Z$.
- We call $N_{Z/X} = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$ the *normal bundle.*
- We have a short exact sequence:

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_X \longrightarrow \Omega_Z \longrightarrow 0$$

To see this, note that locally on $U \subseteq X$ open affine trivializing the locally free sheaves, we get an exact sequence

$$\mathcal{O}_U^{\dim X - \dim Z} \xrightarrow{\alpha} \mathcal{O}_U^{\dim X} \xrightarrow{\beta} \mathcal{O}_U^{\dim Z} \longrightarrow 0$$

We want to check that we get a short exact sequence if we add a 0 on the left, i.e. that $\ker \alpha = 0$. For $x \in U$, we can apply the right exact functor $- \otimes \mathcal{O}_U/\mathfrak{m}_x$ $(\cong k)$ to get

$$k^{\dim X - \dim Z} \longrightarrow k^{\dim X} \longrightarrow k^{\dim Z} \longrightarrow 0$$

which is short exact by dimension reasons. So we see that

$$(\ker \alpha)_x \subseteq \mathfrak{m}_x^{\dim X - \dim Z} \subseteq \mathcal{O}_U^{\dim X - \dim Z}.$$

As this holds for all $x \in U$, we get $\ker \alpha = 0$. So the original sequence is short exact.

Corollary 15.0.1. *If $i : Z \hookrightarrow X$ is a closed embedding of smooth irreducible varieties, then*

$$\omega_Z = \det(\Omega_Z) \cong \det(i^*\Omega_X) \otimes \det(N_{Z/X}) = i^*\omega_X \otimes \det(N_{Z/X}).$$

15.4 Graded Modules

Remark. On an affine variety X, there is a bijection

$$\{\text{finitely generated } A(X)\text{-modules}\}/\cong \longleftrightarrow \{\text{coherent sheaves on } X\}/\cong$$
$$M \longmapsto \widetilde{M}$$
$$\mathcal{F}(X) \longleftarrow\!\shortmid \mathcal{F}.$$

Now we want a similar correspondence for a closed subvariety $X \subseteq \mathbb{P}^n$ with $S(X) = k[x_0, \ldots, x_n]/I(X)$ and graded $S(X)$-modules, though we will only get a bijection up to some further equivalence.

Remark. Let $X \subseteq \mathbb{P}^n$ be a closed subvariety and $S = \bigoplus_{d\in\mathbb{N}} S_d$ its homogeneous coordinate ring.

Definition 15.3. A *graded S-module M* is an S-module with a direct sum decomposition

$$M = \bigoplus_{e\in\mathbb{Z}} M_e$$

as abelian groups, such that $S_d \cdot M_e \subseteq M_{d+e}$. A *morphism* of graded S-modules

$$\phi : M \longrightarrow N$$

is a morphism of S-modules which respects the grading, i.e. $\phi(M_d) \subseteq N_d$ for all $d \in \mathbb{Z}$.

Example 15.3.1. We have the following:

1. If M, N are graded S-modules, then so are $M \oplus N$ and $M \otimes_S N$, with
$$(M \oplus N)_d = M_d \oplus N_d \quad \text{and} \quad (M \otimes_S N)_d = \bigoplus_{e+f=d} (M_e \otimes_{S_0} N_f).$$
2. For $m \in \mathbb{Z}$, let $S(m)$ be the graded S-module such that $S(m)_d = S_{m+d}$.
3. If $\phi : M \to N$ is a morphism of graded S-modules, then $\ker\phi$, $\operatorname{im}\phi$, $\operatorname{coker}\phi$ are graded S-modules.

Exercise 15.1. If M is a finitely generated graded S-module, then show that there exists an exact sequence of graded S-modules of the form

$$\bigoplus_{i=1}^{s} S(a_i) \xrightarrow{\ \alpha\ } \bigoplus_{j=1}^{r} S(b_j) \xrightarrow{\ \beta\ } M \longrightarrow 0$$

Hint: Choose generators $m_1, \ldots, m_r \in M$ that are homogeneous, set $b_j = -\deg m_j$, and let $\beta(e_j) = m_j$.

Remark. Next time, we will define $\widetilde{M} \in \mathrm{QCoh}_X$, and we will see that $\widetilde{S(m)} \cong \mathcal{O}_X(m)$.

Lecture 16

Mar. 5 — Coherent Sheaves on Projective Varieties

16.1 Coherent Sheaves on Projective Varieties

Remark. Let $X \subseteq \mathbb{P}^n$ be a projective variety (i.e. a closed subvariety), and let

$$S = S(X) = k[x_0, \ldots, x_n]/I(X),$$

which is a graded ring. Recall the notion of a graded S-module (Definition 15.3).

Example 16.0.1. Trivially, S is a graded S-module.

Example 16.0.2. If M is a graded S-module, then $M(m)$ is the graded S-module with

$$M(m)_d = M_{d+m}.$$

Remark. Gr-Mod$_S$ is an abelian category.

Remark. We want a correspondence

$$\text{(finitely generated) graded } S\text{-modules} \rightleftarrows \text{(coherent) quasicoherent sheaves on } X$$

To begin with, we want S to correspond to $\mathcal{O}_X$.

Example 16.0.3. Let $X = \mathbb{P}^n$, then $S = k[x_0, \ldots, x_n]$. Note that

$$\mathcal{O}_{\mathbb{P}^n}(D(x_i)) = k[x_0/x_i, \ldots, x_n/x_i] = ((k[x_0, \ldots, x_n])_{x_i})_0 = \left\{ \frac{f}{x_i^d} : f \in k[x_0, \ldots, x_n]_d \right\}$$

Note that $k[x_0, \ldots, x_n]_{x_i}$ is naturally graded. More generally, for $f \in S^+$ of homogeneous degree d, let

$$S_{(f)} := (S_f)_0 = \text{degree 0 part of } S_f = \left\{ \frac{g}{f^i} \in S_f : g \in S_{di},\, i \in \mathbb{N} \right\}.$$

Then there is a homomorphism

$$\begin{aligned} \phi : S_{(f)} &\longrightarrow \mathcal{O}_X(D(f)) \\ g/f^i &\longmapsto (x \mapsto g(x)/f^i(x)), \end{aligned}$$

which we claim is an isomorphism. To see this, consider $Y = C(X) \subseteq \mathbb{A}^{n+1}$, which satisfies $A(Y) = S$. Now observe that we have a commutative diagram

$$\begin{tikzcd} S_{(f)} \arrow[r, "\phi"] \arrow[d, hook] & \mathcal{O}_X(D_X(f)) \arrow[d, hook] \\ S_f \arrow[r, "\cong"] & \mathcal{O}_Y(D_Y(f)) \end{tikzcd}$$

Thus ϕ is injective. Furthermore, if $\psi \in \mathcal{O}_X(D(f))$, we get

$$\psi = g/f^i, \quad g \in S(X).$$

Then $g(\lambda x) = \lambda^{di} g(x)$ for all $x \in X$ such that $g(x) \neq 0$. So $g \in S_{di}$ (assuming the previous condition holds on a dense subset of X, e.g. if g is nonzero on every irreducible component).

Remark. Fix $M \in \text{Gr-Mod}_S$ and $f \in S^+$ homogeneous of degree d. We want an $\mathcal{O}_X$-module. Note that

$$M_{(f)} := \text{degree 0 part of } M_f = \left\{ \frac{m}{f^i} \in M_f : m \in M_{di},\, i \in \mathbb{N} \right\}$$

is an $S_{(f)}$-module. Then we can set

- $\widetilde{M}(D(f)) = M_{(f)}$;
- For $f, g \in S^+$ homogeneous, we have natural maps $M_{(f)} \to M_{(fg)}$ and $M_{(g)} \to M_{(fg)}$, which we can view as the restriction maps $\widetilde{M}(D(f)) \to \widetilde{M}(D(fg))$ and $\widetilde{M}(D(g)) \to \widetilde{M}(D(fg))$.

Then we claim that $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules on the basis $\{D(f) : f \in S^+ \text{ homogeneous}\}$, hence it induces $\widetilde{M} \in \text{Mod}_{\mathcal{O}_X}$. The proof is similar to the affine case.

Proposition 16.1. *$\widetilde{S(m)}$ is an invertible sheaf. In particular, it is quasicoherent.*

Proof. Write $X = D(x_0) \cup \cdots \cup D(x_n)$. Now we have an isomorphism

$$\begin{aligned} \widetilde{S(m)}(D(x_i)) = (S_{x_i})_m &\xrightarrow{\cong} (S_{x_i})_0 \\ r &\longmapsto x_i^{-m} r \\ x_i^m r &\longleftarrow\!\shortmid\, r. \end{aligned}$$

Note that $D(x_i)$ has a basis by $\{D(x_i g) : g \in S^+ \text{ homogeneous}\}$, and

$$\begin{aligned} \widetilde{S(m)}(D(x_i g)) = (S_{x_i g})_m &\xrightarrow{\cong} (S_{x_i g})_0 \\ r &\longmapsto x_i^{-m} r \end{aligned}$$

As this isomorphism is compatible with the previous one, we get that $\widetilde{S(m)}|_{D(x_i)} \cong \mathcal{O}_{D(x_i)}$. So we see that $\widetilde{S(m)}$ is an invertible sheaf. $\square$

Remark. We have $\mathcal{O}_X(m) \cong \widetilde{S(m)}$. To see this, note that we have an isomorphism

$$\begin{aligned} \alpha_i : \widetilde{S(m)}|_{D(x_i)} &\longrightarrow \mathcal{O}_{D(x_i)} \\ x_i^m &\longmapsto 1. \end{aligned}$$

Now observe that we have the composition

$$\begin{array}{ccccc} \mathcal{O}_{D(x_ix_j)} & \xrightarrow{\alpha_j^{-1}} & \widetilde{S(m)}|_{D(x_ix_j)} & \xrightarrow{\alpha_i} & \mathcal{O}_{D(x_ix_j)} \\ 1 & \longmapsto & x_j^m = (x_i^m)(x_j/x_i)^m & \longmapsto & (x_j/x_i)^m \end{array}$$

so the transition functions are $g_{i,j} = (x_j/x_i)^m$. Thus we see that $\widetilde{S(m)} \cong \mathcal{O}_X(m)$.

Remark. Consider the map $\text{Gr-Mod}_S \to \text{Mod}_{\mathcal{O}_X}$ given by the construction $M \mapsto \widetilde{M}$ above. Then:

1. This is a functor.
2. This is exact (localization is exact and the $D(f)$ form a basis).
3. If $M_i \in \text{Gr-Mod}_S$ for $i \in I$, then $\widetilde{\bigoplus_{i\in I} M_i} \cong \bigoplus_{i\in I} \widetilde{M_i}$ (localization commutes with direct sums).

Proposition 16.2. *For $M \in \text{Gr-Mod}_S$, $\widetilde{M}$ is quasicoherent. Furthermore, if M is finitely generated, then $\widetilde{M}$ is coherent.*

Proof. We have an exact sequence

$$\textstyle\bigoplus_i S(-a_i) \longrightarrow \bigoplus_j S(-b_j) \longrightarrow M \longrightarrow 0 \qquad (*)$$

Applying $M \mapsto \widetilde{M}$ (which is exact and commutes with direct sums), we get an exact sequence

$$\textstyle\bigoplus_i \mathcal{O}_X(-a_i) \longrightarrow \bigoplus_j \mathcal{O}_X(-b_j) \longrightarrow \widetilde{M} \longrightarrow 0$$

This gives that $\widetilde{M}$ is quasicoherent. If M is finitely generated, then we can choose the direct sums in $(*)$ to be finite, hence $\widetilde{M}$ is coherent in this case. $\square$

Remark. We have $\widetilde{M} \otimes \mathcal{O}_X(m) \cong \widetilde{M(m)}$. To see this, we can note that

$$(\widetilde{M} \otimes \mathcal{O}_X(m))(D(x_i)) = (M_{x_i})_0 \otimes_{(S_{x_i})_0} (S_{x_i})_m \cong (M_{x_i})_m = \widetilde{M(m)}(D(x_i)).$$

Alternatively, shifting the exact sequence $\bigoplus_i S(-a_i) \to \bigoplus_j S(-b_j) \to M \to 0$ by m (which preserves exactness),

$$\textstyle\bigoplus_i S(m-a_i) \longrightarrow \bigoplus_j S(m-b_j) \longrightarrow M(m) \longrightarrow 0.$$

Applying $M \mapsto \widetilde{M}$, we get that

$$\widetilde{M(m)} = \operatorname{coker}\Big(\bigoplus_i \mathcal{O}_X(m-a_i) \to \bigoplus_j \mathcal{O}_X(m-b_j)\Big) = \widetilde{M} \otimes \mathcal{O}_X(m).$$

Remark. When is $\widetilde{M} = 0$? Clearly this is true if $M = 0$, but there are more examples. Observe that we can write $s/x_i^d = x_i^c s/x_i^{d+c} \in M_{x_i}$, so if $x_i^c s = 0$ for some $c > 0$, then s/x_i^d is zero in $\widetilde{M}(D(x_i))$.

In particular, if there exists $c > 0$ such that $M_i = 0$ for all $i \geq c$, then $\widetilde{M}(D(x_i)) = 0$, so $\widetilde{M} = 0$.

Corollary 16.0.1. *If $\alpha : M \to M'$ is a morphism of graded S-modules and $M_i \to M'_i$ is an isomorphism for $i \gg 0$, then $\widetilde{M} \to \widetilde{M'}$ is an isomorphism.*

Proof. Consider the exact sequence of graded modules

$$0 \longrightarrow \ker\alpha \longrightarrow M \xrightarrow{\alpha} M' \longrightarrow \operatorname{coker}\alpha \longrightarrow 0$$

Now $\ker\alpha_i = 0 = \operatorname{coker}\alpha_i$ for $i \gg 0$. So we get an exact sequence

$$0 \longrightarrow 0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{M'} \longrightarrow 0 \longrightarrow 0$$

Hence $\widetilde{M} \to \widetilde{M'}$ is an isomorphism. □

16.2 Graded Modules from Coherent Sheaves

Remark. How do we get a functor $\Phi : \mathrm{QCoh}_X \to \text{Gr-Mod}_S$?

Example 16.0.4. For $X = \mathbb{P}^n$, we want

$$\Phi(\mathcal{O}_{\mathbb{P}^n}) = k[x_0, \ldots, x_n] = \bigoplus_{d\in\mathbb{N}} k[x_0, \ldots, x_n]_d = \bigoplus_{d\in\mathbb{N}} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)).$$

Definition 16.1. For $\mathcal{F} \in \mathrm{QCoh}_X$, set (below $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}_X(m)$)

$$\Gamma_*(X, \mathcal{F}) = \bigoplus_{m\in\mathbb{Z}} \Gamma(X, \mathcal{F}(m)).$$

Remark. For $M \in \text{Gr-Mod}_S$, we have a morphism of abelian groups

$$M \longrightarrow \Gamma_*(X, \widetilde{M})$$

given by the natural maps $M_i \to \Gamma(X, \widetilde{M(i)}) \subseteq \Gamma_*(X, \widetilde{M})$. In particular, we get

$$S \longrightarrow \Gamma_*(X, \mathcal{O}_X).$$

Using this, we can get a map

$$\begin{array}{ccc} S_m \times \Gamma(X, \mathcal{F}(n)) & \longrightarrow & \Gamma(X, \mathcal{O}_X(m)) \times \Gamma(X, \mathcal{F}(n)) \\ & & \downarrow \\ & & \Gamma(X, \mathcal{F}(m+n)) \end{array}$$

which we can think of as a multiplication. Thus $\Gamma_*(X, \mathcal{F})$ has the structure of a graded S-module.

Proposition 16.3. *For $\mathcal{F} \in \mathrm{QCoh}_X$, there exists a natural isomorphism*

$$\mathcal{F} \xrightarrow{\cong} \widetilde{\Gamma_*(X, \mathcal{F})}.$$

Proof. See Mustata 11.1.18. □

Remark. The functors $(\sim, \Gamma_*)$ form an adjoint pair.

Corollary 16.0.2. *If $\mathcal{F} \in \mathrm{Coh}_X$, then there exists a finitely generated S-module M such that $\mathcal{F} = \widetilde{M}$.*

Proof. Set $N = \Gamma_*(X, \mathcal{F})$ and choose $M \subseteq N$ finitely generated with $\widetilde{M} \cong \mathcal{F}$. □

Lecture 17

Mar. 10 — Sheaf Cohomology

17.1 Motivation for Sheaf Cohomology

Theorem 17.1. *If $\mathcal{G}$ is a coherent sheaf on a projective variety X, then $\Gamma(X, \mathcal{G})$ is a finite dimensional k-vector space.*

Remark. Theorem 17.1 will be a kind of motivating theorem for sheaf cohomology. We will try to give a simple proof. To do this, we use the following strategy:

1. There is a closed embedding $i : X \hookrightarrow \mathbb{P}^n$. Now $\Gamma(X, \mathcal{G}) = \Gamma(\mathbb{P}^n, i_*\mathcal{G})$. Note that $i_*\mathcal{G}$ is coherent since i is a closed embedding. So we reduce to the case $X = \mathbb{P}^n$.

2. When $\mathcal{G} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i)$, Theorem 17.1 holds as
$$\Gamma\Big(\mathbb{P}^n, \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i)\Big) = \bigoplus_{i=1}^r k[x_0, \dots, x_n]_{a_i}.$$

3. We want to reduce to Case (2). By a homework problem, there is a surjection
$$\mathcal{F} = \textstyle\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{G} \longrightarrow 0$$
for some $a_1, \dots, a_r$. Set $\mathcal{E} = \ker(\mathcal{F} \to \mathcal{G})$, then we get a short exact sequence
$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$
Now we get an exact sequence
$$0 \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{E}) \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{F}) \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{G})$$
as taking global sections is left exact. Note that $\Gamma(\mathbb{P}^n, \mathcal{F})$ is finite dimensional. Though we do not always get surjectivity onto $\Gamma(\mathbb{P}^n, \mathcal{G})$, we can construct functors
$$\begin{aligned} \mathrm{QCoh}_X &\longrightarrow \mathrm{Vect}_k \\ \mathcal{F} &\longmapsto H^i(X, \mathcal{F}) \end{aligned}$$
for $i \geq 0$ such that:
   (i) $H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$,
   (ii) $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$,

(iii) given a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ in QCoh_X, there is a long exact sequence

$$H^i(X, \mathcal{E}) \longrightarrow H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{G}) \longrightarrow H^{i+1}(X, \mathcal{E}) \longrightarrow \cdots$$

(iv) $H^i(\mathbb{P}^n, \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}^n}(a_j))$ is a finite dimensional k-vector space.

Then to prove the theorem, it suffices to show that for any coherent sheaf $\mathcal{G}$ on $\mathbb{P}^n$ and $i \geq 0$,

$$H^i(\mathbb{P}^n, \mathcal{G})$$

is a finite dimensional k-vector space. The theorem follows from this by property (i).

Assuming we have H^i satisfying (i)-(iv), we can prove the above claim via downwards induction. As the base case, note that for $i > n$, we have $H^i(\mathbb{P}^n, \mathcal{G}) = 0$ for all quasicoherent sheaves $\mathcal{G}$ on $\mathbb{P}^n$ by (ii). For the inductive step, assume the claim holds for all coherent $\mathcal{G}$ and $i > i_0$. Consider

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} = \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}^n}(a_j) \longrightarrow \mathcal{G} \longrightarrow 0$$

and the corresponding long exact sequence (by (iii))

$$H^{i_0}(\mathbb{P}^n, \mathcal{E}) \longrightarrow H^{i_0}(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^{i_0}(\mathbb{P}^n, \mathcal{G}) \longrightarrow H^{i_0+1}(\mathbb{P}^n, \mathcal{E}) \longrightarrow \cdots$$

Note that $H^{i_0}(X, \mathcal{F})$ is finite dimensional by (iv) and $H^{i_0+1}(X, \mathcal{E})$ is finite dimensional by the inductive hypothesis. So $H^{i_0}(X, \mathcal{G})$ is finite dimensional, and by induction we get the full result.

Thus our goal is to give a definition of H^i, which we will do using the Cech complex.

17.2 Homological Algebra

Definition 17.1. Let R be a ring. A *cochain complex* of R-modules is a sequence of R-modules

$$A = A^\bullet := (A^0 \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^2 \xrightarrow{\partial} \cdots)$$

with morphisms $\partial^i : A^i \to A^{i+1}$ such that $\partial^{i+1} \circ \partial^i = 0$.

Definition 17.2. Let $A^\bullet$ be a cochain complex. The *cohomology* of $A^\bullet$ is

$$H^i(A^\bullet) = \frac{\ker \partial^i}{\operatorname{im} \partial^{i-1}}.$$

We call $Z^i(A^\bullet) = \ker \partial^i$ the *cocycles* and $B^i(A^\bullet) = \operatorname{im} \partial^{i-1}$ the *coboundaries*.

Definition 17.3. A *morphism of cochain complexes* $A^\bullet \to B^\bullet$ is the data of morphisms $\phi^i : A^i \to B^i$ such that the following diagram commutes for all i:

$$\begin{tikzcd} A^i \arrow[r, "\partial"] \arrow[d, "\phi^i"'] & A^{i+1} \arrow[d, "\phi^{i+1}"] \\ B^i \arrow[r, "\partial"] & B^{i+1} \end{tikzcd}$$

Remark. Given a morphism of cochain complexes $A^\bullet \to B^\bullet$, there are induced maps

$$Z^i(A^\bullet) \to Z^i(B^\bullet) \quad \text{and} \quad B^i(A^\bullet) \to B^i(B^\bullet).$$

In particular, we get an induced map on cohomology $H^i(A^\bullet) \to H^i(B^\bullet)$.

Definition 17.4. A *short exact sequence of cochain complexes*

$$0 \longrightarrow A^\bullet \xrightarrow{\phi} B^\bullet \xrightarrow{\psi} C^\bullet \longrightarrow 0$$

is given by morphisms ϕ and ψ such that

$$0 \longrightarrow A^i \xrightarrow{\phi^i} B^i \xrightarrow{\psi^i} C^i \longrightarrow 0$$

is exact for all $i \geq 0$.

Remark. The category of cochain complexes of R-modules is an abelian category.

Proposition 17.1. *Given a short exact sequence of cochain complexes* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$, *we get a long exact sequence in cohomology*

$$H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow \cdots$$

Proof. The difficulty is to define the map $H^{i-1}(C^\bullet) \to H^i(A^\bullet)$. Consider the diagram

$$\begin{tikzcd}
 & H^{i-1}(A) \arrow[r] \arrow[d] & H^{i-1}(B) \arrow[r] \arrow[d] & H^{i-1}(C) \arrow[d] & \\
 & A^i/B^i(A) \arrow[r] \arrow[d, "\partial"] & B^i/B^i(B) \arrow[r] \arrow[d, "\partial"] & C^i/B^i(C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^i(A) \arrow[r] \arrow[d] & Z^i(B) \arrow[r] \arrow[d] & Z^i(C) \arrow[d] & \\
 & H^i(A) \arrow[r] & H^i(B) \arrow[r] & H^i(C) &
\end{tikzcd}$$

The snake lemma then gives a map $H^{i-1}(C) \to H^i(A)$. □

17.3 Cech Cohomology

Definition 17.5. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X, and $\mathcal{U} = \{U_i\}_{i=0}^n$ an open cover of X. For $I \subseteq \{0, \ldots, n\}$, denote $U_I = \bigcap_{i \in I} U_i$. For $p \geq 0$, set

$$C^p(\mathcal{U}, X) = \prod_{\substack{I \subseteq \{0,\ldots,n\} \\ |I| = p+1}} \mathcal{F}(U_I).$$

Then the *Cech complex* is the cochain complex

$$C^\bullet(\mathcal{U}, \mathcal{F}) = [C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} \cdots]$$

with $d^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ given by $(s_I)_I \mapsto (s_J)_J$, where $J = \{i_0 < i_1 < \cdots < i_{p+1}\}$ and

$$s_J = \sum (-1)^k s_{J \setminus \{i_k\}}|_{U_J}.$$

Exercise 17.1. Check that the Cech complex is in fact a cochain complex, i.e. that $d^{i+1} \circ d^i = 0$.

Definition 17.6. The *Cech cohomology* is given by $H^i(\mathcal{U}, \mathcal{F}) = H^i(C^\bullet(\mathcal{U}, \mathcal{F}))$.

Example 17.6.1. We compute $H^0(\mathcal{U}, \mathcal{F})$. We have

$$\begin{aligned} H^0(\mathcal{U}, \mathcal{F}) &= \ker(C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})) \\ &= \ker\Big(\prod_{i \in I} \mathcal{F}(U_i) \to \prod_{0 \le i < j \le n} \mathcal{F}(U_{i,j})\Big) \cong \mathcal{F}(X) = \Gamma(X, \mathcal{F}) \end{aligned}$$

where the isomorphism is by the sheaf axiom.

Example 17.6.2. Let $X = \mathbb{P}^1$ and $\mathcal{U} = (U_0, U_1)$ be the standard open cover.

(i) Let $\mathcal{F} = \mathcal{O}_X$. Then we have

$$C^\bullet(\mathcal{U}, \mathcal{O}_X) = [\mathcal{O}_{\mathbb{P}^1}(U_0) \oplus \mathcal{O}_{\mathbb{P}^1}(U_1) \to \mathcal{O}_{\mathbb{P}_1}(U_0 \cap U_1)] = [k[y/x] \oplus k[x/y] \to k[x/y, y/x]],$$

where the map $\alpha : k[y/x] \oplus k[x/y] \to k[x/y, y/x]$ is given by $(f, g) \mapsto f - g$. Thus

$$H^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = \ker \alpha \cong k \cong \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}).$$

As the map α is surjective, we have that

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{coker} \alpha = 0.$$

(ii) Let $\mathcal{F} = \Omega_{\mathbb{P}^1}$. Then we have the cochain complex

$$\begin{aligned} C^\bullet(\mathcal{U}, \Omega_{\mathbb{P}^1}) &= [\Omega_{\mathbb{P}^1}(U_0) \oplus \Omega_{\mathbb{P}^1}(U_1) \to \Omega_{\mathbb{P}^1}(U_{0,1})] \\ &= [k[y/x]d(y/x) \oplus k[x/y]d(x/y) \to d(y/x)k[x/y, y/x]]. \end{aligned}$$

If we set $s = y/x$, then the above becomes

$$C^\bullet(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = [k[s]ds \oplus k[s^{-1}]ds^{-1} \to k[s^{\pm 1}]ds],$$

where the map $\beta : k[s]ds \oplus k[s^{-1}]ds^{-1} \to k[s^{\pm 1}]ds$ is given by

$$(f(s)ds, g(s^{-1})ds^{-1}) \longmapsto f(s)ds - g(s^{-1})\left(-\frac{1}{s^2}\right)ds.$$

Thus we see that

$$\begin{aligned} H^0(\mathcal{U}, \Omega_{\mathbb{P}^1}) &= \ker \beta = 0 \\ H^1(\mathcal{U}, \Omega_{\mathbb{P}^1}) &= \operatorname{coker} \beta = \operatorname{span}\{\overline{(1/s)ds}\} \cong k. \end{aligned}$$

Example 17.6.3. Let X be a variety and $\mathcal{F} = \mathcal{O}_X^\times$ (this is a sheaf of groups under multiplication). Fix an open cover $\mathcal{U} = \{U_i\}_{i=0}^n$ of X. Consider

$$H^1(\mathcal{U}, \mathcal{O}_X^\times) = \{(g_{i,j})_{i,j} : g_{i,j} \in \mathcal{O}_X^\times(U_{i,j}) \text{ and } g_{i,j}g_{j,k}g_{i,k}^{-1} = 1\}/\sim,$$

where $(g_{i,j})_{i,j} \sim (g'_{i,j})_{i,j}$ if there exists $h_i \in \mathcal{O}_X^\times(U_i)$ such that $h_i^{-1}g_{i,j}h_j = g'_{i,j}$. Thus

$$H^1(\mathcal{U}, \mathcal{O}_X^\times) \cong \{\text{line bundles } \mathbb{L} \to X \text{ such that } \mathbb{L}_{U_i} \cong \mathbb{1}_{U_i}\}/\cong.$$

In fact, the above is an isomorphism of groups.

Remark. From now on, we will assume that X is a variety, $\mathcal{F}$ is a quasicoherent sheaf on X, and that $\mathcal{U} = \{U_i\}_{i=0}^n$ is an open affine cover of X.

Proposition 17.2. *We have the following:*

1. $H^i(\mathcal{U}, \mathcal{F}) = 0$ *for* $i > n$, *where* $n = |\mathcal{U}| - 1$.
2. *Given a short exact sequence* $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ *in* QCoh_X, *there is an induced long exact sequence on Cech cohomology:*

$$H^i(\mathcal{U}, \mathcal{E}) \longrightarrow H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{U}, \mathcal{G}) \longrightarrow H^{i+1}(\mathcal{U}, \mathcal{E}) \longrightarrow \cdots$$

3. *Given a closed embedding* $i : Z \hookrightarrow X$ *and a quasicoherent sheaf* $\mathcal{G}$ *on* Z, *there is an isomorphism*

$$H^i(\{U_i \cap Z\}_{i=0}^n, \mathcal{G}) \cong H^i(\mathcal{U}, i_*\mathcal{G}).$$

4. $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Proof. (1) We have $C^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > n$.

(2) Since the U_I are affine, we get a short exact sequence

$$0 \longrightarrow \mathcal{E}(U_I) \longrightarrow \mathcal{F}(U_I) \longrightarrow \mathcal{G}(U_I) \longrightarrow 0$$

for any $I \subseteq \{0, \ldots, n\}$. So we get an induced short exact sequence of Cech complexes

$$0 \longrightarrow C^\bullet(\mathcal{U}, \mathcal{E}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{G}) \longrightarrow 0$$

This gives a long exact sequence on Cech cohomology by Proposition 17.1.

(3) We have $\mathcal{G}(U_I \cap Z) = (i_*\mathcal{G})(U_I)$, so we get $C^\bullet(\mathcal{U} \cap Z, \mathcal{G}) \cong C^\bullet(\mathcal{U}, i_*\mathcal{G})$. $\square$

Lecture 18

Mar. 12 — Sheaf Cohomology, Part 2

18.1 Cech Cohomology, Continued

Remark. Does $H^i(\mathcal{U}, \mathcal{F})$ depend on the choice of the affine open cover? Consider finite open affine covers $\mathcal{U}' \subseteq \mathcal{U} = \{U_i\}_{i=0}^n$ with $\mathcal{U}' = \{U_j\}_{j \in J}$ with $J \subseteq \{0, \ldots, n\}$. Then we get a morphism of chain complexes $C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}', \mathcal{F})$ and hence an induced morphism $\phi^i_{\mathcal{U},\mathcal{U}'} : H^i(\mathcal{U}, \mathcal{F}) \to H^i(\mathcal{U}', \mathcal{F})$.

Theorem 18.1. *The map $\phi^i_{\mathcal{U},\mathcal{U}'}$ is an isomorphism. In particular, $H^i(X, \mathcal{F}) := H^i(\mathcal{U}, \mathcal{F})$ is independent of the choice of finite open affine cover $\mathcal{U}$.*

Proof. If suffices to consider the case when $\mathcal{U} = \{U_i\}_{i=0}^n$ and $\mathcal{U}' = \{U_i\}_{i=1}^n$, and then the rest follows by induction. Consider the short exact sequence of chain complexes

$$\begin{tikzcd}
0 \arrow[d] & & & \\
A^\bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \prod_{\substack{|I|=p+1 \\ 0 \in I}} \mathcal{F}(U_I) \arrow[r] & \cdots \\
C^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \prod_{|I|=p+1} \mathcal{F}(U_I) \arrow[r] & \cdots \\
C^\bullet(\mathcal{U}', \mathcal{F}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \prod_{\substack{|I|=p+1 \\ 0 \notin I}} \mathcal{F}(U_I) \arrow[r] & \cdots \\
0 & & &
\end{tikzcd}$$

Then we get a long exact sequence on cohomology

$$H^i(A^\bullet) \longrightarrow H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{U}', \mathcal{F}) \longrightarrow H^{i+1}(A^\bullet)$$

Now let $A^p = C^{p+1}(\{U_i \cap U_0\}_{i=1}^n, \mathcal{F})$, where we note that $\{U_i \cap U_0\}_{i=1}^n$ is an open cover of U_0. We claim that $H^i(A^\bullet)$ for all $i \geq 0$ (Proposition 18.1), so $H^i(\mathcal{U}, \mathcal{F}) \to H^i(\mathcal{U}', \mathcal{F})$ is an isomorphism. $\square$

Remark. When X is not an algebraic variety, one can set

$$H^i(X, \mathcal{F}) = \varinjlim_{\mathcal{U} \text{ open cover}} H^i(\mathcal{U}, \mathcal{F})$$

Proposition 18.1. *If X is an affine variety, $\mathcal{U} = \{U_i\}_{i=0}^n$ is a finite open affine cover, and $\mathcal{F}$ is a quasicoherent sheaf on X, then $H^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We first assume that $U_0 = X$. Then we have a short exact sequence of complexes

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \prod_{\substack{|I|=1 \\ 0\in I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{\substack{|I|=2 \\ 0\in I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r] & \prod_{|I|=1} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=2} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r] & \prod_{\substack{|I|=1 \\ 0\notin I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{\substack{|I|=2 \\ 0\notin I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
 & & 0 & 0 &
\end{tikzcd}$$

Note that $\prod_{|I|=p+1,0\in I} \mathcal{F}(U_I) = \mathcal{F}(X)$, and we see that the first row is the last row shifted right by 1. Calling the middle row by $A^\bullet$ and using the long exact sequence on cohomology, we see that $H^i(A^\bullet) = 0$ for all i. Thus we see that $H^i(\mathcal{U}, \mathcal{F}) = 0$ for $i > 0$. This proves the special case.

Now consider the general case. Let $A = \mathcal{O}_X(X)$. Note that $C^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of A-modules Now $H^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$ if and only if $C^\bullet(\mathcal{U}, \mathcal{F})$ is exact in degree $p > 0$. By commutative algebra, this is equivalent to there being $f_1, \ldots, f_r \in A$ such that $D(f_1) \cup \cdots \cup D(f_r) = X$ and $C^\bullet(\mathcal{U}, \mathcal{F})_{f_i}$ is exact in degree $p > 0$. Then we observe that

$$C^p(\mathcal{U}, \mathcal{F})_{f_i} = \prod_{|I|=p+1} \mathcal{F}(U_I)_{f_i} = \prod_{|I|=p+1} \mathcal{F}(U_I \cap D(f_i)).$$

Now choose $f_1, \ldots, f_r \in A$ such that $D(f_1) \cup \cdots \cup D(f_r) = X$ and for each i, there exists j_i such that $D(f_i) \subseteq U_{j_i}$. Then

$$H^p(C^\bullet(\mathcal{U}, \mathcal{F})_{f_i}) = H^p(\{U_j \cap D(f_i)\}_{j=0}^n, \mathcal{F}) = 0$$

for $p > 0$ by the special case. So $C^\bullet(\mathcal{U}, \mathcal{F})$ is exact in degree $p > 0$. □

Remark. Theorem 18.1 implies the following:

- For a quasicoherent sheaf $\mathcal{F}$, we can define $H^i(X, \mathcal{F}) := H^i(\mathcal{U}, \mathcal{F})$ where $\mathcal{U}$ is any finite open affine cover, and $H^i(X, \mathcal{F})$ is independent of the choice of $\mathcal{U}$.
- If X is affine and $\mathcal{F}$ is quasicoherent, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Theorem 18.2 (Serre)**.** *If X is a variety that is not affine, then there exists a quasicoherent sheaf $\mathcal{F}$ on X such that $H^i(X, \mathcal{F}) \neq 0$ for some $i > 0$.*

Lecture 19

Mar. 17 — Sheaf Cohomology, Part 3

19.1 Examples of Sheaf Cohomology

Remark. We want to compute $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ and use it to give applications.

Example 19.0.1. We have already seen in Example 17.6.2 that

$$H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \begin{cases} k & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

and that (note that we have $\Omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$)

$$H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = \begin{cases} k & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 19.0.2. We have seen in homework that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = k[x_0, \ldots, x_n]_m$.

Example 19.0.3. Let $X = \mathbb{P}^n$, $\mathcal{U} = \{U_i\}_{i=0}^n$ for $U_i = D(x_i)$, and $S = \bigoplus_{m \geq 0} S_m = \bigoplus_{m \geq 0} k[x_0, \ldots, x_n]_m$. We want to compute H^i of the complex

$$\textstyle\prod_i \mathcal{O}_{\mathbb{P}^n}(m)(U_i) \longrightarrow \prod_{i<j} \mathcal{O}_{\mathbb{P}^n}(m)(U_{i,j}) \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(m)(U_{0,\ldots,n})$$

We know that for $U_{i_0,\ldots,i_r} = D(x_{i_0} \cdots x_{i_r})$, we have

$$\mathcal{O}_{\mathbb{P}^n}(m)(U_{i_0,\ldots,i_r}) = (S_{x_{i_0} \cdots x_{i_r}})_m.$$

To make this simpler, write $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(m)$. Now

$$H^i(X, \mathcal{F}) = \bigoplus_{m \in \mathbb{Z}} H^i(X, \mathcal{O}_X(m)),$$

and observe that $\mathcal{F}(U_{i_0,\ldots,i_r}) = S_{x_{i_0} \cdots x_{i_r}}$. So we have

$$C^\bullet(\mathcal{U}, \mathcal{F}) = \left[\prod_i S_{x_i} \longrightarrow \prod_{i<j} S_{x_i x_j} \longrightarrow \cdots \longrightarrow S_{x_0 \cdots x_n} \right].$$

Then we want to compute $H^i(C^\bullet(\mathcal{U}, \mathcal{F}))$ and keep track of the grading. For $i = 0$,

$$H^0(X, \mathcal{F}) = \ker \left(\prod_i S_{x_i} \longrightarrow \prod_{i<j} S_{x_i x_j} \right) = S,$$

so $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = S_m$. In particular, $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for $m < 0$. For $i = n$,

$$H^n(X, \mathcal{F}) = \operatorname{coker}\left(\prod_i S_{x_0 \cdots \widehat{x}_i \cdots x_n} \longrightarrow S_{x_0 \cdots x_n}\right) = \bigoplus_{a_i < 0} k\overline{x_0^{a_0} \cdots x_n^{a_n}}.$$

So $H^n(X, \mathcal{O}_{\mathbb{P}^n}(m)) = \bigoplus_{a_i<0, \sum a_i = m} k\overline{x_0^{a_0} \cdots x_n^{a_n}}$. In particular, $H^n(X, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for $m \geq -n$ and

$$H^n(X, \mathcal{O}_{\mathbb{P}^n}(-n-1)) = k\overline{x_0^{-1} \cdots x_n^{-1}}.$$

Observe that there is a perfect pairing

$$\begin{aligned} H^0(X, \mathcal{O}_{\mathbb{P}^n}(m)) \times H^n(X, \mathcal{O}_{\mathbb{P}^n}(-m-n-1)) &\longrightarrow H^n(X, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \cong k \\ (x_0^{a_0} \cdots x_n^{a_n}, \overline{x_0^{b_0} \cdots x_n^{b_n}}) &\longmapsto \overline{x_0^{a_0+b_0} \cdots x_n^{a_n+b_n}}. \end{aligned}$$

where $a_i \geq 0$ and $\sum a_i = m$, and $b_i < 0$ and $\sum b_i = -m-n-1$. Note that

$$\overline{x_0^{a_0+b_0} \cdots x_n^{a_n+b_n}} = \begin{cases} \overline{x_0^{-1} \cdots x_n^{-1}} & \text{if } a_i + b_i = -1 \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

This says that $H^n(X, \mathcal{O}_{\mathbb{P}^n}(-m) \otimes \omega_{\mathbb{P}^n}) \cong H^0(X, \mathcal{O}_{\mathbb{P}^n}(m))^\vee$, which is a special case of *Serre duality.*

Now for the middle cohomology, we claim that $H^i(X, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for $0 < i < n$. For this, it suffices to show that $H^i(X, \mathcal{F}) = 0$ for $0 < i < n$. We argue by induction. Consider the short exact sequence

$$0 \longrightarrow S(-1) \xrightarrow{\times x_n} S \longrightarrow S/x_nS \longrightarrow 0$$

Then taking $M \mapsto \widetilde{M}$ gives a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_H \longrightarrow 0$$

with $H = V(x_n) \cong \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$. Set $\mathcal{F}_H = i_*(\bigoplus_{m\in\mathbb{Z}} \mathcal{O}_H(m))$. Applying $- \otimes \mathcal{F}$ gives

$$0 \longrightarrow \mathcal{F}(-1) \xrightarrow{\times x_n} \mathcal{F} \longrightarrow \mathcal{F}_H \longrightarrow 0$$

This gives a long exact sequence

$$\begin{aligned} &H^0(X, \mathcal{F}(-1)) \xhookrightarrow{\times x_n} H^0(X, \mathcal{F}) \twoheadrightarrow H^0(X, \mathcal{F}_H) \longrightarrow \cdots \\ &\longrightarrow H^i(X, \mathcal{F}(-1)) \xrightarrow{\times x_n} H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F}_H) \longrightarrow \cdots \\ &\longrightarrow H^{n-1}(X, \mathcal{F}(H)) \xhookrightarrow{\delta} H^n(X, \mathcal{F}(-1)) \xrightarrow{\times x_n} H^n(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F}_H) \end{aligned}$$

Note that $H^i(X, \mathcal{F}_H) = 0$ for $1 \leq i < n-1$ by induction. One can check that δ is multiplication by $1/x_n$, so it is injective. Thus for $0 < i < n$, we have an isomorphism (note that $\mathcal{F}(-1) \cong \mathcal{F}$)

$$H^i(X, \mathcal{F}(-1)) \xrightarrow{\times x_n} H^i(X, \mathcal{F}).$$

So $H^i(X, \mathcal{F})$, viewed as an S-module, satisfies that $x_n : H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$ is an isomorphism. Now

$$H^i(X, \mathcal{F})_{x_n} \cong H^i(C^\bullet(\mathcal{U}, \mathcal{F}))_{x_n} \cong H^i(C^\bullet(\{U_i \cap U_n\}_{i=0}^n, \mathcal{F}|_{U_n})) = 0$$

as $\{U_i \cap U_n\}_{i=0}^n$ is an open cover of U_n, and U_n is an affine variety. So $H^i(X, \mathcal{F}) = 0$ for $0 < i < n$. This then implies that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for $0 < i < n$. From the duality perspective, we have

$$H^i(X, \mathcal{O}_{\mathbb{P}^n}(m)) \cong H^{n-i}(X, \mathcal{O}_{\mathbb{P}^n}(m)^\vee \otimes \omega_{\mathbb{P}^n})^\vee.$$

19.2 Applications to Coherent Sheaves

Remark. For this section, let $X \hookrightarrow \mathbb{P}^n$ be a projective variety and $\mathcal{F}$ a quasicoherent sheaf on X.

Proposition 19.1. *$H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.*

Proof. Let $r = \dim X$, and choose hyperplanes $H_0, \ldots, H_r$ on X such that $X \cap (H_0 \cap \cdots \cap H_r) = \varnothing$. So

$$X = U_0 \cup \cdots \cup U_r$$

with $U_i = X \setminus H_i$. As the U_i are affine, we get $H^i(X, \mathcal{F}) = H^i(C^\bullet(\{U_i\}_{i=0}^r, \mathcal{F})) = 0$ for $i > r$. □

Proposition 19.2. *If $\mathcal{F}$ is coherent, then $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space.*

Proof. See the reduction given in Section 17.1, where the only missing part was that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ is finite dimensional for $i \geq 0$ and $m \in \mathbb{Z}$. □

Theorem 19.1 (Serre vanishing)**.** *If $\mathcal{F}$ is coherent, then there exists $m_0 = m_0(\mathcal{F})$ such that*

$$H^i(X, \mathcal{F}(m)) = 0$$

for $i > 0$ and $m \geq m_0$.

Proof. Note that $i_*(\mathcal{F}(m)) \cong (i_*\mathcal{F})(m)$, and we have

$$H^i(X, \mathcal{F}(m)) \cong H^i(\mathbb{P}^n, i_*\mathcal{F}(m)),$$

so we may reduce to the case $X = \mathbb{P}^n$. Then there is a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_X(a_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

Twisting by $\mathcal{O}_X(m)$ gives

$$0 \longrightarrow \mathcal{E}(m) \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_X(m + a_i) \longrightarrow \mathcal{F}(m) \longrightarrow 0$$

So we get a long exact sequence on cohomology

$$H^i(\mathbb{P}^n, \bigoplus_{i=1}^r \mathcal{O}_X(m + a_i)) \longrightarrow H^i(\mathbb{P}^n, \mathcal{F}(m)) \longrightarrow H^{i+1}(\mathbb{P}^n, \mathcal{E}(m)) \longrightarrow \cdots$$

Now $H^{i+1}(\mathbb{P}^n, \mathcal{E}(m)) = 0$ by induction and $H^i(\mathbb{P}^n, \bigoplus_{i=1}^r \mathcal{O}_X(m + a_i)) = 0$ for $m > -n - 1 - a_i$. So we see that $H^i(\mathbb{P}^n, \mathcal{F}(m)) = 0$ for $i > 0$ and $m \gg 0$. □

Remark. The coherent assumption in Theorem 19.1 is necessary: Let $X = \mathbb{P}^n$ and $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(m)$.

Remark. We can get a cohomological characterization of ampleness. One can check that for an invertible sheaf $\mathcal{L}$ on a projective variety X, the following are equivalent:

1. $\mathcal{L}$ is ample;
2. $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated for any coherent sheaf $\mathcal{F}$ and $m \gg 0$ (dependent on $\mathcal{F}$);
3. $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for any coherent $\mathcal{F}$ on X, $i > 0$, and $m \gg 0$ (dependent on $\mathcal{F}$).

Note that in general (not necessarily for a projective variety), the definition of ampleness is given by (2).

Lecture 20

Mar. 19 — Sheaf Cohomology, Part 4

20.1 More Examples of Sheaf Cohomology

Remark. Last time, we computed that for $X = \mathbb{P}^n$ and $m \in \mathbb{Z}$,

1. $H^0(X, \mathcal{O}_X(m)) = k[x_1, \dots, x_n]_m$,
2. $H^i(X, \mathcal{O}_X(m)) = 0$ for $i \neq 0, n$,
3. there is a perfect pairing

$$H^i(X, \mathcal{O}_X(m)) \times H^{n-i}(X, \mathcal{O}_X(-m-n-1)) \longrightarrow H^n(X, \mathcal{O}_X(-n-1)) \cong k.$$

We can also write $\omega_{\mathbb{P}^n}$ for $\mathcal{O}_{\mathbb{P}^n}(-n-1)$ (and $\omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^m}(m)^\vee$) for $\mathcal{O}_{\mathbb{P}^n}(-m-n-1)$). We can make an isomorphism $H^n(X, \omega_{\mathbb{P}^n}) \cong k$ by choosing the Cech cocycle class

$$\frac{x_0^n}{x_1 \cdots x_n} d(x_1/x_0) \wedge \cdots \wedge d(x_n/x_0)$$

uniquely determined by $U_0, \dots, U_n$. For example, $[x_0 : \cdots : x_n] \mapsto [a_0 x_0 : \cdots : a_n x_n]$ fixes this class.

Example 20.0.1. Let D be a degree d hypersurface in $\mathbb{P}^n$ with $n \geq 2$. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_D(m) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(m) \longrightarrow \mathcal{O}_D(m) \longrightarrow 0$$

Note that $\mathcal{I}_D \cong \mathcal{O}_{\mathbb{P}^n}(-D) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$. We get a long exact sequence on cohomology

$$\cdots \longrightarrow H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)) \longrightarrow H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \longrightarrow H^i(D, \mathcal{O}_D(m))$$

$$\longrightarrow H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)) \longrightarrow \cdots$$

From this we can conclude that:

1. $i = 0$: We have

$$H^0(D, \mathcal{O}_D(m)) \cong \operatorname{coker}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)) \xrightarrow{\times f} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))) \cong S_m / f S_{m-d}$$

   with $S = k[x_0, \dots, x_n]$.
2. $0 < i < n-1$: $H^i(D, \mathcal{O}_D(m)) = 0$ since $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)) = 0$.

3. $i = n-1$: $H^{n-1}(D, \mathcal{O}_D(m)) = \ker(H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)) \twoheadrightarrow H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)))$.

Example 20.0.2. Let C be a smooth degree d plane curve in $\mathbb{P}^2$. Then $H^0(C, \mathcal{O}_C) = k$ and

$$\begin{aligned}\dim H^1(C, \mathcal{O}_C) &= \dim\ker(H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})) = \dim H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \\ &= \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) = \binom{2+d-3}{2} = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}.\end{aligned}$$

Note that if $k = \mathbb{C}$, then $(d-1)(d-2)/2$ is the genus of the Riemann surface $C^{\mathrm{an}} \subseteq \mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}$.

Example 20.0.3. $H^\bullet(X, \mathcal{O}_X)$ can have "middle" cohomology: The *Kunneth formula* says that

$$H^i(X \times Y, p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}) = \bigoplus_{j+k=i} H^j(X, \mathcal{F}) \otimes H^k(Y, \mathcal{G}).$$

For example, we can take $X = Y = C$ to be a smooth degree d plane curve and $\mathcal{F} = \mathcal{G} = \mathcal{O}_C$, with

$$p_1^*\mathcal{F} \otimes p_2^*\mathcal{G} \cong \mathcal{O}_{C\times C}.$$

Theorem 20.1 (Serre duality)**.** *If X is a smooth projective variety of pure dimension n and $\mathcal{F}$ is a locally free sheaf on X, then $H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee$.*

Exercise 20.1. Check that Serre duality holds for $\mathbb{P}^n$ and hypersurfaces.

Remark. Serre duality implies in particular that $H^n(X, \mathcal{F}) \cong H^0(X, \mathcal{F}^\vee \otimes \omega_X)$.

20.2 Invariants from Sheaf Cohomology

Definition 20.1. Let X be a projective variety and $\mathcal{F}$ a coherent sheaf on X. Thus $H^i(X, \mathcal{F})$ is a finite-dimensional k-vector space. Then we define:

1. $h^i(X, \mathcal{F}) = \dim H^i(X, \mathcal{F})$,
2. the *Euler characteristic*
$$\chi(X, \mathcal{F}) = \sum_{i\geq 0} (-1)^i \dim H^i(X, \mathcal{F}) \in \mathbb{Z}.$$

Lemma 20.1. *If $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ is a short exact sequence of coherent sheaves on a projective variety X, then*

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{E}) + \chi(X, \mathcal{G}).$$

Proof. Take the long exact sequence on cohomology

$$H^i(X, \mathcal{E}) \longrightarrow H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{G}) \longrightarrow H^{i+1}(X, \mathcal{E}) \longrightarrow \cdots$$

Then by linear algebra, $\sum_i (-1)^i h^i(X, \mathcal{F}) = \sum_i (-1)^i h^i(X, \mathcal{E}) + \sum_i (-1)^i h^i(X, \mathcal{G})$. $\square$

Example 20.1.1. We have

$$\chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \dim H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) + (-1)^n \dim H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m-n-1)).$$

Then we see that:

- If $m \geq -n$, this is $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \binom{n+m}{n} = ((m+1)\cdots(m+n))/n!$.
- If $m \leq -n-1$, this is

$$(-1)^n h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m-n-1)) = (-1)^n \binom{-m-1}{n} = (-1)^n \frac{(-m-1)\cdots(-m-n)}{n!} = \frac{(m+1)\cdots(m+n)}{n!}.$$

Thus $\chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = ((m+1)\cdots(m+n))/n!$ for all $m \in \mathbb{Z}$.

Theorem 20.2. *If $\mathcal{F}$ is a coherent sheaf on a projective variety X, then*

$$P_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$$

is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F}$.

Lemma 20.2. *If $\phi : \mathbb{Z} \to \mathbb{Z}$ is a function such that there exists $Q \in \mathbb{Q}[t]$ with*

$$\phi(m) - \phi(m-1) = Q(m)$$

for all $m \in \mathbb{Z}$, then there exists $P \in \mathbb{Q}[t]$ such that $\phi(m) = P(m)$ for all $m \in \mathbb{Z}$. Furthermore, we have $\deg P = \deg Q + 1$.

Proof. See the notes from Algebraic Geometry I. □

Definition 20.2. Let A be a Noetherian ring and M a finitely generated A-module. An *associated prime* of M is a prime $\mathfrak{p} \leq A$ such that $\mathfrak{p} = \operatorname{Ann}(m)$ for some $0 \neq m \in M$. Let

$$\operatorname{Ass}(M) = \{\mathfrak{p} \leq A : \mathfrak{p} \text{ is an associated prime of } M\}.$$

Theorem 20.3. *We have the following:*

1. $\operatorname{Ass}(M)$ *is finite.*
2. *If $M \neq 0$, then* $\operatorname{Ass}(M) \neq 0$.
3. $\{$*zero divisors of* $M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$.

Proposition 20.1. *If $\mathcal{F}$ is a nonzero coherent sheaf on $\mathbb{P}^n$, then for a general hyperplane $H \subseteq \mathbb{P}^n$, there is a short exact sequence*

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(-H) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_H \longrightarrow 0$$

and $\dim \operatorname{Supp}(\mathcal{F} \otimes \mathcal{O}_H) = \dim \operatorname{Supp} \mathcal{F} - 1$.

Proof. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-H) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_H \longrightarrow 0$$

Applying $- \otimes \mathcal{F}$, we get an exact sequence

$$\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(-H) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_H \longrightarrow 0$$

So it suffices to determine when the map $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(-H) \to \mathcal{F}$ is injective. We can check this on each $U_i = D(x_i) \subseteq \mathbb{P}^n$: Write

(i) $M_i = \mathcal{F}(U_i)$, which is a finitely generated module over $A_i = k[x_0/x_i, \ldots, x_n/x_i]$;

(ii) $H = V(\sum a_j x_j)$ and $h_i = (a_0/a_i)(x_0/x_i) + \cdots + (a_n/a_i)(x_n/x_i) \in A_i$.

Choose H general so that $h_i \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M_i)} \mathfrak{p}$. Then the map

$$\begin{aligned} M_i \cong M_i \otimes h_i A_i &\longrightarrow M_i \\ m &\longmapsto m h_i \end{aligned}$$

is injective. So we have that

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(-H) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_H \longrightarrow 0$$

is a short exact sequence. Check as an exercise that $\mathrm{Supp}(\mathcal{F} \otimes \mathcal{O}_H) = \mathrm{Supp}(\mathcal{F}) \cap H$. $\square$

Proof of Theorem 20.2. First check that we may reduce to the case $X = \mathbb{P}^n$: If $i : X \hookrightarrow \mathbb{P}^n$ is a closed embedding, then we have

$$\chi(X, \mathcal{F}(m)) = \chi(\mathbb{P}^n, i_*\mathcal{F}(m)).$$

We proceed by induction on $\dim \mathrm{Supp}\, \mathcal{F}$: First note that if $\dim \mathrm{Supp}\, \mathcal{F} = 0$, then $\mathcal{F} \otimes \mathcal{O}_H = 0$. Now by Proposition 20.1, we have a short exact sequence

$$0 \longrightarrow \mathcal{F}(m-1) \longrightarrow \mathcal{F}(m) \longrightarrow \mathcal{F}(m) \otimes \mathcal{O}_H \longrightarrow 0$$

where $\dim \mathrm{Supp}(\mathcal{F}(m) \otimes \mathcal{O}_H) = \dim \mathrm{Supp}\, \mathcal{F} - 1$. Thus we have

$$\chi(X, \mathcal{F}(m)) - \chi(X, \mathcal{F}(m-1)) = \chi(X, \mathcal{F}(m) \otimes \mathcal{O}_H),$$

where $\chi(X, \mathcal{F}(m) \otimes \mathcal{O}_H)$ is a polynomial of degree $\dim \mathrm{Supp}\, \mathcal{F} - 1$ by the induction hypothesis. Thus by Lemma 20.2, $\chi(X, \mathcal{F}(m))$ is a polynomial of degree $\dim \mathrm{Supp}\, \mathcal{F}$. $\square$

Remark. Let $X \subseteq \mathbb{P}^n$ be a projective variety and consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_X(m) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(m) \longrightarrow \mathcal{O}_X(m) \longrightarrow 0$$

By Serre vanishing, all of the higher cohomology is 0 for $m \gg 0$. Thus

$$\begin{aligned} \dim S(X)_m = \dim \frac{k[x_0, \ldots, x_n]_m}{I_X(m)} &= h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) - h^0(\mathbb{P}^n, \mathcal{I}_X(m)) \\ &= h^0(X, \mathcal{O}_X(m)) = \chi(X, \mathcal{O}_X(m)), \end{aligned}$$

where the last two equalities hold for $m \gg 0$. Note that $\dim S(X)_m$ is a polynomial for $m \gg 0$, whereas $\chi(X, \mathcal{O}_X(m))$ is a polynomial for all $m \in \mathbb{Z}$.

Lecture 21

Mar. 31 — Riemann-Roch Theorem

21.1 More on Vanishing of Cohomology

Theorem 21.1. *If X is a projective variety and $\mathcal{F}$ is a coherent sheaf on X, then*

$$H^i(X, \mathcal{F}) = 0$$

for all $i > \dim \operatorname{Supp}(\mathcal{F})$.

Proof. Set $r = \dim \operatorname{Supp}(\mathcal{F})$ and choose hyperplanes $H_0, \ldots, H_r$ such that

$$\operatorname{Supp}(\mathcal{F}) \cap H_0 \cap \cdots \cap H_r = \varnothing.$$

Now take $U_i = X \setminus H_i$, which are open affines with

$$(U_0 \cup \cdots \cup U_r) \cap \operatorname{Supp}(\mathcal{F}) = \operatorname{Supp}(\mathcal{F}).$$

Choose an open affine cover $U_{r+1}, \ldots, U_s$ such that $U_{r+1} \cup \cdots \cup U_s = X \setminus \operatorname{Supp}(\mathcal{F})$. So

$$H^i(X, \mathcal{F}) = H^i_{U_0, \ldots, U_s}(X, \mathcal{F}) = H^i_{U_0, \ldots, U_r}(X, \mathcal{F})$$

since $\mathcal{F}(U_i) = 0$ for $i > r$. Thus $H^i(X, \mathcal{F}) = 0$ for $i > r$. $\square$

21.2 Riemann-Roch Theorem

Remark. The *Riemann-Roch problem* is the following: Let C be a smooth projective curve and $\mathcal{L}$ an invertible sheaf on C. What is $h^0(C, \mathcal{L}) = \dim \Gamma(C, \mathcal{L})$? The following are special cases:

- If $\mathcal{L} = \omega_C$, then $h^0(C, \omega_C)$ is the dimension of the space of algebraic 1-forms.
- If $\mathcal{L} = \omega_C(D) = \omega_C \otimes \mathcal{O}_C(D)$, then $h^0(C, \omega_C(D))$ gives the dimension of the space of algebraic 1-forms with certain conditions on poles.
- If $\mathcal{L} = \mathcal{O}_C$, then $h^0(C, \mathcal{O}_C) = 1$.

We can approach this problem as follows:

- instead compute $\chi(C, \mathcal{L}) = h^0(C, \mathcal{L}) - h^1(C, \mathcal{L})$;
- using $\operatorname{Cl}(C) \xrightarrow{\cong} \operatorname{Pic}(C)$ given by $[D] \mapsto \mathcal{O}_C(D)$, we may assume that $\mathcal{L} = \mathcal{O}_C(D)$.

Theorem 21.2 (Riemann-Roch)**.** *If C is a smooth projective curve and D is a divisor on C, then*

$$\chi(C, \mathcal{O}_C(D)) = \deg D + 1 - g,$$

where $g = h^1(C, \mathcal{O}_C)$ is the genus *of C, and* $\deg D = \sum_{i=1}^r a_i$ *where $D = \sum_{i=1}^r a_i P_i$.*

Remark. By Serre duality, we can rewrite

$$\begin{aligned}\chi(C, \mathcal{O}_C(D)) &= h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) \\ &= h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(-D) \otimes \omega_C) \\ &= h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(K_C - D)),\end{aligned}$$

where K_C is the canonical divisor of C. Riemann showed (1857) that

$$h^0(C, \mathcal{O}_C(D)) \geq \deg D + 1 - g,$$

and Roch improved this (1865) to the full statement of Theorem 21.2:

$$h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(K_C - D)) = \deg D + 1 - g.$$

Proof of Theorem 21.2. We proceed by induction. The base case is $D = 0$, and we have

$$\chi(C, \mathcal{O}_C) = h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = 1 - g = \deg(0) + 1 - g.$$

Now we show the inductive step. It suffices to show that for any $P \in C$, the theorem holds for D if and only if it holds for $D - P$ (then we can repeatedly add or subtract points until we get to the desired divisor). Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-P) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_P \longrightarrow 0,$$

and apply $- \otimes \mathcal{O}_C(D)$ to get a short exact sequence (note that the sheaves above are locally free)

$$\begin{tikzcd} & & \mathcal{O}_P \arrow[d, equal] & \\ 0 \arrow[r] & \mathcal{O}_C(D-P) \arrow[r] & \mathcal{O}_C(D) \arrow[r] & \mathcal{O}_P \otimes \mathcal{O}_C(D) \arrow[r] & 0. \end{tikzcd}$$

Then we have that

$$\begin{aligned}\chi(C, \mathcal{O}_C(D)) - \chi(C, \mathcal{O}_C(D-P)) &= \chi(C, \mathcal{O}_P) = h^0(C, \mathcal{O}_P) - h^1(C, \mathcal{O}_P) \\ &= h^0(C, \mathcal{O}_P) = 1 = \deg D - \deg(D - P),\end{aligned}$$

so Riemann-Roch holds for D if and only if it holds for $D - P$. $\square$

Lecture 22

Apr. 2 — Applications of Riemann-Roch

22.1 Applications of Riemann-Roch

Example 22.0.1. Let $C = \mathbb{P}^1$. We have $g(\mathbb{P}^1) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. Using the isomorphism

$$\begin{aligned} \mathrm{Cl}(\mathbb{P}^1) &\xrightarrow{\cong} \mathbb{Z} \\ D &\longmapsto \deg D, \end{aligned}$$

we have $\mathcal{O}_{\mathbb{P}^1}(D) = \mathcal{O}_{\mathbb{P}^1}(d)$ with $d = \deg D$. So

$$\chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D)) = \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) - h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = d + 1 = \deg D + 1 - g.$$

Example 22.0.2. Let $C \subseteq \mathbb{P}^2$ be a smooth plane curve of degree d. Then we have seen that

$$g = h^1(C, \mathcal{O}_C) = \frac{(d-1)(d-2)}{2}.$$

Corollary 22.0.1. *If D_1 and D_2 are linearly equivalent divisors on C, then* $\deg D_1 = \deg D_2$.

Proof. If $D_1 \sim D_2$, then $\mathcal{O}_C(D_1) \cong \mathcal{O}_C(D_2)$, so $\chi(C, \mathcal{O}_C(D_1)) = \chi(C, \mathcal{O}_C(D_2))$, which then implies that $\deg D_1 = \deg D_2$ by the Riemann-Roch theorem. $\square$

Definition 22.1. Let $\mathcal{L}$ be an invertible sheaf on C. The *degree* of $\mathcal{L}$ is $\deg \mathcal{L} := \deg D$ where D is a divisor such that $\mathcal{L} \cong \mathcal{O}_C(D)$.

Remark. The degree of $\mathcal{L}$ is well-defined by Corollary 22.0.1.

Example 22.1.1. Let $\mathcal{L}$ be an invertible sheaf on C. If $h^0(C, \mathcal{L}) \geq 1$, then there exists $0 \neq s \in H^0(C, \mathcal{L})$. Thus there exists $D \geq 0$ such that $\mathcal{L} \cong \mathcal{O}_C(D)$, so $\deg \mathcal{L} = \deg D \geq 0$. Furthermore, if $h^0(C, \mathcal{L}) \geq 1$ and $\deg \mathcal{L} = 0$, then $\mathcal{L} \cong \mathcal{O}_C$.

Corollary 22.0.2. $\deg \omega_C = 2g - 2$.

Proof. By Serre duality, we have $\chi(C, \mathcal{O}_C) = -\chi(C, \omega_C)$ (as $h^0(\mathcal{O}_C) = h^1(\omega_C)$ and $h^1(\mathcal{O}_C) = h^0(\omega_C)$). So by the Riemann-Roch theorem, we have

$$-(1-g) = -\chi(C, \mathcal{O}_C) = \chi(C, \omega_C) = \deg \omega_C + 1 - g,$$

so $\deg \omega_C = 2g - 2$, as desired. $\square$

Remark. If $0 \neq \eta \in \Gamma(C, \omega_C)$, then $\omega_C \cong \mathcal{O}_C(\operatorname{div}_C \eta)$, so by Corollary 22.0.2,

$$2g - 2 = \deg \operatorname{div}_C \eta = \# \text{ zeros of } \eta.$$

Definition 22.2. We say that a divisor D on C is *non-special* if $h^0(C, \mathcal{O}_C(K - D)) = 0$ (or equivalently, by Serre duality, if $h^1(C, \mathcal{O}_C(D)) = 0$).

Remark. For D non-special, we have $h^0(C, \mathcal{O}_C(D)) = \deg D + 1 - g$ by Riemann-Roch.

Lemma 22.1. *If $\deg D \geq 2g - 1$, then D is non-special.*

Proof. If $\deg D \geq 2g - 1$, then $\deg(K - D) \leq (2g - 2) - (2g - 1) = -1$, so $h^0(C, \mathcal{O}_C(K - D)) = 0$. □

Remark. If $\deg D \geq 2g - 1$, then D is non-special, which implies $h^0(C, \mathcal{O}_C(D)) = \deg D + 1 - g$.

22.2 More on Embeddings to Projective Space

Proposition 22.1. *An invertible sheaf $\mathcal{L}$ on C is globally generated if and only if*

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-P)) = 1$$

for every $P \in C$.

Remark. We define $\mathcal{L}(-P) := \mathcal{L} \otimes \mathcal{O}_C(-P)$. In these cases, the fibre will be denoted by

$$\mathcal{L}_{(P)} := \mathcal{L}_P / \mathfrak{m}_P \mathcal{L}_P = H^0(C, \mathcal{L}/\mathcal{I}_P \mathcal{L}).$$

Proof of Proposition 22.1. For $P \in C$, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-P) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_P \longrightarrow 0$$

Applying $- \otimes \mathcal{L}$, we get an exact sequence (since $\mathcal{L}$ is invertible)

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_P \longrightarrow 0.$$

Taking H^0, we then get an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{L}(-P)) & \longrightarrow & H^0(\mathcal{L}) & \xrightarrow{\phi} & H^0(\mathcal{L} \otimes \mathcal{O}_P) \\ & & & & & & \| \\ & & & & & & \mathcal{L}_{(P)} \cong k \end{array}$$

So $\mathcal{L}$ is globally generated at P if and only if ϕ is surjective, if and only if $h^0(\mathcal{L}) - h^0(\mathcal{L}(-P)) = 1$. □

Proposition 22.2. *An invertible sheaf $\mathcal{L}$ on C is very ample if and only if*

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-P - Q)) = 2$$

for every $P, Q \in C$ (not necessarily distinct).

Proof. By the proof of Proposition 22.1, we have

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-P)) = 1$$

for all $P \in C$, so $\mathcal{L}$ is globally generated. In this setting, we know that $\mathcal{L}$ is very ample if and only if $\mathcal{L}$ separates points and tangent vectors. For $P \neq Q \in C$, consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-P-Q) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_P \oplus \mathcal{O}_Q \longrightarrow 0$$

Applying $- \otimes \mathcal{L}$ and H^0, we get an exact sequence

$$0 \longrightarrow H^0(\mathcal{L}(-P-Q)) \longrightarrow H^0(\mathcal{L}) \xrightarrow{\psi} \mathcal{L}_{(P)} \oplus \mathcal{L}_{(Q)}$$

So $\mathcal{L}$ separates P, Q if and only if ψ is surjective, if and only if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-P-Q)) = 2$.

Next fix $P \in C$, and consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-2P) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C/\mathcal{O}_C(-2P) \longrightarrow 0$$

Again applying $- \otimes \mathcal{L}$ and H^0, we get an exact sequence

$$0 \longrightarrow H^0(\mathcal{L}(-2P)) \longrightarrow H^0(\mathcal{L}) \xrightarrow{\gamma} \mathcal{L}_P/\mathfrak{m}_P^2\mathcal{L}_P$$

(note that $\mathcal{O}_C/\mathcal{O}_C(-2P) = i_P(\mathcal{O}_{C,P}/\mathfrak{m}_P^2)$). Thus $\mathcal{L}$ separates tangent vectors at P if and only if γ is surjective, if and only if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2P)) = 2$. $\square$

Corollary 22.0.3. *We have the following:*

1. *if* $\deg \mathcal{L} \geq 2g$, *then* $\mathcal{L}$ *is globally generated;*
2. *if* $\deg \mathcal{L} \geq 2g+1$, *then* $\mathcal{L}$ *is very ample.*

Proof. (1) If $\deg \mathcal{L} \geq 2g$, then

$$h^0(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g \quad \text{and} \quad h^0(C, \mathcal{L}(-P)) = \deg \mathcal{L}(-P) + 1 - g$$

as $\mathcal{L}$ and $\mathcal{L}(-P)$ are both non-special. So $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-P)) = 1$ for all $P \in C$, so $\mathcal{L}$ is globally generated by Proposition 22.1.

(2) The proof is similar, use that $\deg \mathcal{L} \geq 2g+1$ implies

$$\deg \mathcal{L} > \deg \mathcal{L}(-P) \geq \deg \mathcal{L}(-P-Q) \geq 2g-1,$$

so $\mathcal{L}, \mathcal{L}(-P), \mathcal{L}(-P-Q)$ are all non-special. So

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-P-Q)) = \deg \mathcal{L} - \deg \mathcal{L}(-P-Q) = 2,$$

so $\mathcal{L}$ is very ample by Proposition 22.2. $\square$

Remark. Corollary 22.0.3 is sharp: Let $C = \mathbb{P}^1$, then $\deg \mathcal{O}_{\mathbb{P}^1} = 2g$, but $\mathcal{O}_{\mathbb{P}^1}$ is not very ample. Similarly, $\deg \mathcal{O}_{\mathbb{P}^1}(-1) = 2g-1$, but $\mathcal{O}_{\mathbb{P}^1}(-1)$ is not globally generated.

22.3 Applications to Curves

Corollary 22.0.4. *An invertible sheaf $\mathcal{L}$ on C is ample if and only if $\deg \mathcal{L} > 0$.*

Proof. ($\Rightarrow$) If $\mathcal{L}$ is ample, then there exists $m > 0$ such that $\mathcal{L}^{\otimes m}$ is very ample, so

$$m \deg \mathcal{L} = \deg \mathcal{L}^{\otimes m} > 0,$$

which implies that $\deg \mathcal{L} > 0$.

($\Leftarrow$) If $\deg \mathcal{L} > 0$, then $\deg \mathcal{L}^{\otimes(2g+1)} = (\deg \mathcal{L})(2g+1) \geq 2g+1$, so $\mathcal{L}^{\otimes(2g+1)}$ is very ample. $\square$

Corollary 22.0.5. *If the genus of C is $g = 0$, then $C \cong \mathbb{P}^1$.*

Proof. Fix $P \in C$ and set $\mathcal{L} = \mathcal{O}_C(P)$. Since $\deg \mathcal{L} = 1 \geq 2g+1$, we know $\mathcal{L}$ is very ample. Also,

$$h^0(\mathcal{L}) = \deg \mathcal{L} + 1 - g = 1 + 1 - 0 = 2.$$

So we get an embedding $C \hookrightarrow \mathbb{P}^{2-1} = \mathbb{P}^1$. Since $\dim C = \dim \mathbb{P}^1 = 1$ and $\mathbb{P}^1$ is irreducible, $C \cong \mathbb{P}^1$. $\square$

Corollary 22.0.6. *If the genus of C is $g = 1$, then C is isomorphic to a cubic curve in $\mathbb{P}^2$.*

Proof. Fix $P \in C$ and set $\mathcal{L} = \mathcal{O}_C(3P)$. Since $\deg \mathcal{L} = 3 \geq 2g+1$, we know $\mathcal{L}$ is very ample. Similarly,

$$h^0(\mathcal{L}) = \deg \mathcal{L} + 1 - g = 3.$$

Thus we get an embedding $C \hookrightarrow \mathbb{P}^{3-1} = \mathbb{P}^2$. So C is a plane curve of some degree d. Using that

$$1 = g = \frac{1}{2}(d-1)(d-2),$$

we see that we can only have $d = 3$. $\square$

Lecture 23

Apr. 7 — Classification of Curves

23.1 Higher Genus Curves

Remark. Recall that a smooth plane curve $C_d \subseteq \mathbb{P}^2$ of degree d has genus $(d-1)(d-2)/2$. This does not cover most $g \geq 2$. One may also try to consider complete intersections, but again this does not cover everything. The full answer is the following:

1. For $g \geq 2$, there exists an irreducible quasiprojective variety $\mathcal{M}_g$ of dimension $3g-3$ parametrizing all smooth projective genus g curves.
2. If $C \in \mathcal{M}_g$, then either
   - there exists a 2-to-1 cover $C \to \mathbb{P}^1$ (*hyperelliptic curves*), or
   - $\mathcal{O}_C(K_C)$ is very ample and induces an embedding $C \hookrightarrow \mathbb{P}^{g-1}$ (*canonically embedded curves*).

To show (2) (we will not prove (1)), we need to understand *finite morphisms*.

23.2 Finite Morphisms

Definition 23.1. A morphism of varieties $f : X \to Y$ is *finite* if there exists an open affine cover $\{U_i\}$ of Y such that $V_i = f^{-1}(U_i)$ is affine and $\mathcal{O}_Y(U_i) \to \mathcal{O}_X(V_i)$ is a finite morphism of rings (i.e. $\mathcal{O}_X(V_i)$ is a finitely generated $\mathcal{O}_Y(U_i)$-module).

Example 23.1.1. Let $d > 0$ and consider the morphism

$$\begin{aligned} \mathbb{A}^1_x &\longrightarrow \mathbb{A}^1_y \\ x &\longmapsto x^d. \end{aligned}$$

This morphism is finite, as $k[y] \to k[x]$ is given by $y \mapsto x^d$, which is finite (note that $k[x]$ is generated as a $k[y]$-module by $1, x, \ldots, x^{d-1}$).

Remark. We have the following properties of finite morphisms:

1. If $f : X \to Y$ is finite and $U \subseteq Y$ is open affine, then $V = f^{-1}(U)$ is open affine and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is finite. (See Mustata 5.3.2 for a proof.)
2. A closed embedding $Z \hookrightarrow Y$ is finite.

   To see this, note that it suffices to prove this on an open cover, so we may assume Y is affine. Set $A = \mathcal{O}_Y(Y)$. Then $Z = V(I_Z)$ with $I_Z \subseteq A$ and $A = \mathcal{O}_Y(Y) \to \mathcal{O}_Z(Z) = A/I_Z$, which is finite.

3. If $X \to Y$ is finite, then $\#f^{-1}(\{y\}) < \infty$ for any $y \in Y$.

   To see this, note that this is local in a neighborhood of y, so we may assume that Y is affine. Then by (1), X is also affine. Then $B := \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) =: C$ is finite. Now

   $$A(f^{-1}(\{y\})) = C/I_{f^{-1}(\{y\})} = (C/\mathfrak{m}_y C)^{\mathrm{red}} = (C \otimes_B B/\mathfrak{m}_y B)^{\mathrm{red}}$$

   where R^{red} denotes the reduced ring corresponding to R. Now $C \otimes_B B/\mathfrak{m}_y B$ is a finite $B/\mathfrak{m}_y B$-module and $B/\mathfrak{m}_y \cong k$, so we see that $\dim_k A(f^{-1}(\{y\})) < \infty$. Thus it suffices to show:

   Claim. If X is an affine variety and $\dim_k A(X) < \infty$, then X is a finite set of points.

   Proof of claim. We may assume that X is irreducible.[1] Since $\dim_k A(X) < \infty$ and $A(X)$ is a domain, for any $0 \neq f \in A(X)$, we have an injective linear map given by

   $$\begin{aligned} A(X) &\longrightarrow A(X) \\ 1 &\longmapsto f. \end{aligned}$$

   By finite dimensionality, this map is also surjective, so f has an inverse. Thus $A(X)$ is a field, and hence X is a single point. □

Remark. Morphisms with finite fibers are often not finite. For example, the inclusion $\mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$ has finite fibers, but we have $k[x] = A(\mathbb{A}^1) \to A(\mathbb{A}^1 \setminus \{0\}) = k[x^{\pm 1}]$.

Theorem 23.1. *A morphism of complete varieties $f : X \to Y$ with finite fibers is finite.*

Proof. See Mustata 14.1.8. □

23.3 Finite Morphisms of Curves

Remark. Assume $f : X \to Y$ is a surjective morphism of smooth projective curves. Note that such a morphism always has finite fibers (as $\dim X = \dim Y = 1$), so f is finite by Theorem 23.1.

Definition 23.2. Define the *degree* of f to be $\deg f = \deg(K(X)/K(Y))$, where $f^* : K(Y) \hookrightarrow K(X)$. Note that since $\mathrm{tr.deg}_k\, K(Y) = \mathrm{tr.deg}_k\, K(X) = 1$, this extension is finite.

Definition 23.3. We say that f is *separable* if $K(X)/K(Y)$ is separable (recall that this condition always holds when $\mathrm{char}\, k = 0$.)

Example 23.3.1. The morphism $\mathbb{A}^1_{\overline{\mathbb{F}}_p} \to \mathbb{A}^1_{\overline{\mathbb{F}}_p}$ given by $x \to x^p$ (technically its extension to $\mathbb{P}^1_{\overline{\mathbb{F}}_p} \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$) is non-separable.

Proposition 23.1. *If D is a divisor on Y, then $\deg f^*D = (\deg f)(\deg D)$. Here, f^*D is defined by pulling back local equations (recall that every divisor is Cartier when Y is smooth).*

Proof. As the formula is linear in D, we may assume that $D = P$ for some $P \in Y$. Fix an open affine $P \in U \subseteq Y$, and set $V = f^{-1}(U)$. Now consider

$$A = \mathcal{O}_Y(U) \xrightarrow{f^*} \mathcal{O}_X(V) = B,$$

[1] At this point, this is essentially the statement that a finite integral domain is a field, but with a finite-dimensional vector space instead. Either way, the key property is that an injective (linear) map is automatically also surjective.

which is finite by our assumptions. Let $\mathfrak{m}_P \subseteq A$ be the ideal of functions vanishing at P. Now $B_{\mathfrak{m}_P}$ is a finite $A_{\mathfrak{m}_P}$-module, $A_{\mathfrak{m}_P}$ is a DVR, and $B_{\mathfrak{m}_P}$ is torsion-free, so $B_{\mathfrak{m}_P}$ is a finite rank free $A_{\mathfrak{m}_P}$-module. So we have $B_{\mathfrak{m}_P} \cong A_{\mathfrak{m}_P}^{\oplus r}$ for some $r \geq 0$, and $r = d := \deg f$ by localizing away from 0. So

$$d = \dim_k(B_{\mathfrak{m}_P}/\mathfrak{m}_P B_{\mathfrak{m}_P}) = \dim_k H^0(X, \mathcal{O}_{f^*P}) = \chi(X, \mathcal{O}_{f^*P}).$$

Now we can consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-f^*P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{f^*P} \longrightarrow 0$$

from which we see that $\chi(X, \mathcal{O}_{f^*P}) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-f^*P)) = \deg f^*P$ by Riemann-Roch. So

$$\deg f^*P = d = (\deg f)(\deg P)$$

since $\deg P = 1$, which is the desired formula. □

Example 23.3.2. Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ be given by $x \mapsto x^2$. Then

$$f^*([0] + [1]) = 2[0] + [1] + [-1].$$

Remark. How do we compare genus(X) and genus(Y)? We have an exact sequence

$$f^*\Omega_Y \xrightarrow{\alpha} \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0 \tag{$*$}$$

Claim. If f is separable, then $(*)$ is short exact.

Proof. Since $f^*\Omega_Y$ and Ω_X are invertible, it suffices to show that $\alpha \neq 0$. Since f is separable, we know that $\Omega_{K(X)/K(Y)} = 0$. Taking stalks at $p \in X$ and setting $q = f(p)$, we get

$$\Omega_{\mathcal{O}_{Y,q}/k} \otimes \mathcal{O}_{X,p} \xrightarrow{\alpha_p} \Omega_{\mathcal{O}_{X,p}} \longrightarrow \Omega_{\mathcal{O}_{X,p}/\mathcal{O}_{Y,q}} \longrightarrow 0$$

and applying $- \otimes K(X)$ gives

$$\begin{array}{ccccccc} \Omega_{K(Y)/k} \otimes K(X) & \xrightarrow{\alpha_{K(X)}} & \Omega_{K(X)/k} & \longrightarrow & \Omega_{K(X)/K(Y)} & \longrightarrow & 0 \\ & & & & \| & & \\ & & & & 0 & & \end{array}$$

Thus $\alpha_{K(X)} \neq 0$, so $\alpha_P \neq 0$, and hence $\alpha \neq 0$. □

So now we assume that f is separable and study $\Omega_{X/Y}$.

Lecture 24

Apr. 9 — Classification of Curves, Part 2

24.1 Ramification

Remark. As before, let $f : X \to Y$ be a surjective morphism of (smooth projective) curves. We will try to relate $\Omega_{X/Y}$ to "ramification."

Definition 24.1. Fix $P \in X$ and set $Q = f(P)$. Choose uniformizers $u \in \mathcal{O}_{X,P}$ and $v \in \mathcal{O}_{Y,Q}$. Now $f^*v = cu^{e_P}$ for some $c \in \mathcal{O}_{X,P}^\times$ and $e_P \in \mathbb{Z}_{\geq 1}$ (note that v vanishes at Q, so f^*v must vanish at P, hence $e_P \neq 0$). Equivalently, $e_P = \operatorname{coeff}_P(f^*Q)$. We say that f is *ramified* at P if $e_P > 1$.

Example 24.1.1. Let $\mathbb{A}^1 \to \mathbb{A}^1$ be given by $x \mapsto x^d$. Then

$$e_P = \begin{cases} d & \text{if } P = 0, \\ 1 & \text{if } P \neq 0. \end{cases}$$

Definition 24.2. The *ramification divisor* of X is

$$\operatorname{Ram}_f = \sum_{P \in X} (e_P - 1)P.$$

We call $\operatorname{Supp}(\operatorname{Ram}_f)$ the *ramification locus*.

Remark. Assume going forward that f is separable. Note that $(\Omega_X)_P = \mathcal{O}_{X,P}du$ since

$$(\Omega_X)_{(P)} \cong \mathfrak{m}_P/\mathfrak{m}_P^2.$$

Similarly, we have $(f^*\Omega_Y)_P \cong (\Omega_Y)_Q \otimes \mathcal{O}_{X,P}$, generated by $dv \otimes 1$, and $(\mathcal{O}_{\operatorname{Ram}_f})_P \cong \mathcal{O}_{X,P}/u^{e_P-1}\mathcal{O}_{X,P}$. Now we have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (f^*\Omega_Y)_P \arrow[r] & (\Omega_X)_P \arrow[r] & (\Omega_{X/Y})_P \arrow[r] & 0 \\ & dv \otimes 1 \arrow[r, mapsto] & d(cu^{e_P}) & & \end{tikzcd}$$

where $d(cu^{e_P}) = u^{e_P}dc + cu^{e_P-1}du = cu^{e_P-1}du$ since c is a unit. Thus $\Omega_{X/Y} = \mathcal{O}_{\operatorname{Ram}_f}$, so we have

$$0 \longrightarrow f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \mathcal{O}_{\operatorname{Ram}_f} \longrightarrow 0$$

Applying $- \otimes \Omega_X^\vee$, we get a short exact sequence

$$0 \longrightarrow \Omega_X^\vee \otimes f^*\Omega_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\operatorname{Ram}_f} \longrightarrow 0$$

Note that $\mathcal{O}_{\mathrm{Ram}_f}$ is supported at points, so $\Omega_X^\vee \otimes \mathcal{O}_{\mathrm{Ram}_f} \cong \mathcal{O}_{\mathrm{Ram}_f}$. Thus $\Omega_X^\vee \otimes f^*\Omega_Y \cong \mathcal{O}_X(-\mathrm{Ram}_f)$, so

$$\Omega_X \cong f^*\Omega_Y \otimes \mathcal{O}_X(\mathrm{Ram}_f).$$

This implies the following result:

Theorem 24.1 (Riemann-Hurwitz formula). *We have the following:*

1. $\mathrm{Ram}_f + f^*K_Y \sim K_X$,
2. $2g(X) - 2 = (\deg f)(2g(Y) - 2) + \deg \mathrm{Ram}_f$.

Remark. We have the following:

1. $g(X) \geq g(Y)$,
2. $g(X) = g(Y)$ occurs only when $d = 1$ and there is no ramification, or X, Y are either both genus 0 or both genus 1.

24.2 Classification of Higher Genus Curves

Remark. Morally, hyperelliptic curves are "curves with 2-to-1 maps to $\mathbb{P}^1$."

Definition 24.3. The *gonality* of a curve X, denoted $\mathrm{gon}(X)$, is the smallest degree of an invertible sheaf $\mathcal{L}$ on X with $h^0(X, \mathcal{L}) \geq 2$.

Example 24.3.1. We have the following:

1. $\mathrm{gon}(\mathbb{P}^1) = 1$ (computed by $\mathcal{O}_{\mathbb{P}^1}(1)$, or equivalently, $\mathcal{O}_{\mathbb{P}^1}(P)$ for any $P \in \mathbb{P}^1$).
2. $\mathrm{gon}(E) = 2$ for an elliptic curve E, since $\deg \mathcal{L} = h^0(E, \mathcal{L}) - h^0(E, \mathcal{L}^\vee)$, thus $\deg \mathcal{L} = 1$ implies $h^0(E, \mathcal{L}) = 1$ and $\deg \mathcal{L} = 2$ implies $h^0(E, \mathcal{L}) = 2$.

Lemma 24.1. *The gonality* $\mathrm{gon}(X)$ *is the smallest degree of a finite map* $X \to \mathbb{P}^1$.

Proof. If $f : X \to \mathbb{P}^1$ is a degree d map, then $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ has degree d and $h^0(X, \mathcal{L}) \geq 2$, so

$$\deg f \geq \mathrm{gon}(X).$$

Conversely, fix a minimal degree invertible sheaf $\mathcal{L}$ on X with $h^0(X, \mathcal{L}) \geq 2$. Then for $P \in X$,

$$h^0(X, \mathcal{L}(-P)) = h^0(X, \mathcal{L}) - 1,$$

by minimality, and in fact $h^0(X, \mathcal{L}) = 2$ by a similar argument. Thus $\mathcal{L}$ is globally generated and induces a map $f : X \to \mathbb{P}^1$ with $f^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{L}$. Thus $\deg \mathcal{L} = (\deg f)(\deg \mathcal{O}_{\mathbb{P}^1}(1)) = \deg f$. □

Remark. An elliptic curve has a degree 2 map $E \to \mathbb{P}^1$, so by the Riemann-Hurwitz formula,

$$\deg \mathrm{Ram}_f = 4.$$

So there are 4 distinct points of ramification. In fact, the 4 ramification points (up to isomorphisms of $\mathbb{P}^1$) classifies the elliptic curve up to isomorphism.

Definition 24.4. A curve X is *hyperelliptic* if $\mathrm{gon}(X) = 2$ (i.e. the minimal degree map $X \to \mathbb{P}^1$ has degree 2).

Remark. A hyperelliptic curve X with a degree 2 morphism $X \to \mathbb{P}^1$ can be constructed from the data of Ram_f.

Example 24.4.1. Every smooth genus 2 curve is hyperelliptic and has a 2-to-1 map $X \to \mathbb{P}^1$ ramified at 6 points.

To see this, note that if $g = 2$, then $\deg \omega_X = 2 \cdot 2 - 2 = 2$. Now

$$h^0(X, \omega_X) = h^1(X, \omega_X) + \deg \omega_X + 1 - 2 = h^0(X, \mathcal{O}_X) + 1 = 2,$$

so $\mathrm{gon}(X) \leq 2$. But $\mathrm{gon}(X) = 1$ would imply $X \cong \mathbb{P}^1$, so $\mathrm{gon}(X) = 2$.

Remark. For $g \geq 2$, we have that ω_X is globally generated.

To see this, it suffices to show that $h^0(X, \omega_X) - h^0(X, \omega_X(-P)) = 1$ for all $P \in X$. We have

$$h^0(X, \omega_X(-P)) = h^1(X, \mathcal{O}_X(P)) = h^0(X, \mathcal{O}_X(P)) - \deg P + g - 1,$$

where the first equality is by Serre duality and the second equality is by Riemann-Roch. Now we must have $h^0(X, \mathcal{O}_X(P)) = 1$ as otherwise $h^0(X, \mathcal{O}_X(P)) \geq 2$ would imply $\mathrm{gon}(X) = 1$. Thus

$$h^0(X, \omega_X(-P)) = g - 1 = h^1(X, \mathcal{O}_X) - 1 = h^0(X, \omega_X) - 1,$$

where the last equality is by Serre duality. Thus ω_X is globally generated.

Remark. For $g \geq 2$, when is ω_X very ample? Note that ω_X is not very ample if and only if there exist $P, Q \in X$ such that

$$h^0(X, \omega_X) - h^0(X, \omega_X(-P - Q)) \neq 2.$$

Since we already know ω_X is globally generated, this is equivalent to saying that

$$h^0(X, \omega_X) - h^0(X, \omega_X(-P - Q)) = 1.$$

By Serre duality, this equivalent to there being $P, Q \in X$ such that

$$g - h^1(X, \mathcal{O}_X(P + Q)) = 1,$$

which by Riemann-Roch is equivalent to there being $P, Q \in X$ such that

$$g - (h^0(X, \mathcal{O}_X(P + Q)) - \deg(P + Q) + g - 1) = 1,$$

which is equivalent to $h^0(X, \mathcal{O}_X(P + Q)) = 2$. This is then equivalent to $\mathrm{gon}(X) = 2$. This proves:

Theorem 24.2. *For a curve X with $g \geq 2$, either X is hyperelliptic or ω_X is very ample.*

Definition 24.5. A *canonical curve* X is a curve of genus ≥ 2 with ω_X very ample.

Remark. If X is a canonical curve, then there exists an embedding $X \hookrightarrow \mathbb{P}^{g-1}$ induced by ω_X.

Example 24.5.1. The following are some classifications of genus g canonical curves:

1. $g = 3$: quartic curve in $\mathbb{P}^2$,
2. $g = 4$: intersection of a quadric with a cubic in $\mathbb{P}^3$,
3. $g = 5$: intersection of 3 quadrics in $\mathbb{P}^4$,
4. $g = 6$: intersection of $G(2, 5)$ in $\mathbb{P}^9$ with a linear subspace $\mathbb{P}^5 \subseteq \mathbb{P}^9$ and a quadric.

Lecture 25

Apr. 21 — Surfaces

25.1 Intersection Numbers

Remark. Recall that if X is a projective variety and $\mathcal{F} \in \mathrm{Coh}_X$, then

$$\chi(X, \mathcal{F} \otimes \mathcal{O}_X(m))$$

(the *Hilbert polynomial*) is a polynomial of degree $\dim \mathrm{Supp}(\mathcal{F})$ in m. When $\dim X = 1$, we have an explicit description (by Riemann-Roch) as follows:

Theorem 25.1. *If C is a curve and D is a Cartier divisor on C, then*

$$\chi(C, \mathcal{O}_C(mD)) = m \deg D + 1 - g(C).$$

How does this generalize to higher dimensions?

Let $X \subseteq \mathbb{P}^n$ be a smooth projective surface and D a divisor on X. We want to extract something like "$\deg D$." The idea is the following: Take a general hyperplane $H \subseteq \mathbb{P}^n$, and consider $\#(D \cap H)$. Note

$$D \cap H = D \cap (H \cap X),$$

where D and $H \cap X$ are both curves.

Theorem 25.2. *There exists a symmetric bilinear pairing*

$$\begin{aligned} \mathrm{Div}\, X \times \mathrm{Div}\, X &\longrightarrow \mathbb{Z} \\ (C, D) &\longmapsto C \cdot D \end{aligned}$$

such that

1. *(linear equivalence) if $C, D, C', D' \in \mathrm{Div}\, X$ with $C \sim C'$ and $D \sim D'$, then $C \cdot D = C' \cdot D'$;*
2. *if C, D are distinct irreducible curves, then $C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p$, where*

$$(C \cdot D)_p = \text{local intersection number at } p = \dim_k(\mathcal{O}_{X,p}/(f, g))$$

with $I_{C,p} = (f)$ and $I_{D,p} = (g)$.

Remark. If C and D are smooth curves meeting transversely at p (i.e. the tangent directions of C, D at p are distinct), then $(C \cdot D)_p = 1$.

To see this, note that C, D meet transversely at p if and only if $k[f] \neq k[g]$ in $\mathfrak{m}_p/\mathfrak{m}_p^2$, which happens if and only if $[f], [g]$ generate $\mathfrak{m}_p/\mathfrak{m}_p^2$. By Nakayama's lemma, this happens if and only if f, g generate $\mathfrak{m}_p$, i.e. $(f, g) = \mathfrak{m}_p$. This implies that $\dim_k(\mathcal{O}_{X,p}/(f, g)) = \dim_k k = 1$.

Remark. The pairing $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ induces a symmetric bilinear form

$$\begin{aligned} \mathrm{Cl}(X) \times \mathrm{Cl}(X) &\longrightarrow \mathbb{Z} \\ ([C],[D]) &\longmapsto C \cdot D. \end{aligned}$$

Example 25.0.1. Consider $\mathbb{P}^2$, where $\mathrm{Cl}(\mathbb{P}^2) = \mathbb{Z}[H]$ for some line $H \subseteq \mathbb{P}^2$. Then

$$H \cdot H = H \cdot H' = 1,$$

where H' is a line different from H in $\mathbb{P}^2$. So if $C, D \subseteq \mathbb{P}^2$ are plane curves of degrees c, d, then

$$C \cdot D = (cH) \cdot (dH) = cdH^2 = cd.$$

This recovers the version of Bezout's theorem which counts multiplicity.

Example 25.0.2. Consider $\mathbb{P}^1_{x_0:x_1} \times \mathbb{P}^1_{y_0:y_1}$, where $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2]$ for

$$H_1 = \{[0:1]\} \times \mathbb{P}^1 \quad \text{and} \quad H_2 = \mathbb{P}^1 \times \{[0:1]\}.$$

Then $H_1 \cdot H_1 = H_1 \cdot ([1:0] \times \mathbb{P}^1) = 0$ (similarly, $H_2 \cdot H_2 = 0$) and $H_1 \cdot H_2 = 1$.

Example 25.0.3. Let $\pi : B_p\mathbb{P}^2 \to \mathbb{P}^2$ be the blowup and $E \subseteq B_p\mathbb{P}^2$ the exceptional divisor. Then

$$\mathrm{Cl}(B_p\mathbb{P}^2) = \mathbb{Z}[E] \oplus \mathbb{Z}[\widetilde{H}],$$

where $p \notin H \subseteq \mathbb{P}^2$ is a hyperplane. Here $\widetilde{H}$ denotes the strict transform. Then we can compute:

1. $\widetilde{H} \cdot \widetilde{H}$. Choose a line $p \notin H' \subseteq \mathbb{P}^2$ different from H. Now $H \sim H'$, so

$$\widetilde{H} = \pi^* H \sim \pi^* H' = \widetilde{H}',$$

   and thus $\widetilde{H} \cdot \widetilde{H} = \widetilde{H} \cdot \widetilde{H}' = 1$.

2. $E \cdot \widetilde{H} = 0$.

3. $E \cdot E$. Choose distinct lines $p \in L_i \subseteq \mathbb{P}^2$ for $i = 1, 2$. Now $L_1 \sim L_2 \sim H$, and

$$\pi^* L_i = E + \widetilde{L}_i.$$

   So $E \sim \pi^* L_i - \widetilde{L}_i \sim \widetilde{H} - \widetilde{L}_i$, and thus

$$\begin{aligned} E \cdot E = (\widetilde{H} - \widetilde{L}_1) \cdot (\widetilde{H} - \widetilde{L}_2) &= (\widetilde{H} \cdot \widetilde{H}) - (\widetilde{L}_1 \cdot \widetilde{H}) - (\widetilde{L}_2 \cdot \widetilde{H}) + (\widetilde{L}_1 \cdot \widetilde{L}_2) \\ &= 1 - (\widetilde{L}_1 \cdot \widetilde{H}) - (\widetilde{L}_2 \cdot \widetilde{H}) + (\widetilde{L}_1 \cdot \widetilde{L}_2). \end{aligned}$$

   Note that $\widetilde{L}_1$ and $\widetilde{L}_2$ do not intersect in the blowup, while $\widetilde{L}_i$ and $\widetilde{H}$ intersect at a point. Thus

$$E \cdot E = 1 - 1 - 1 + 0 = -1.$$

Lemma 25.1. *If C and D are distinct curves on X with C smooth and irreducible, then*

$$C \cdot D = \deg(\mathcal{O}_X(D)|_C).$$

Proof. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

and applying $- \otimes \mathcal{O}_C$ gives a short exact sequence (recall that $\mathcal{O}_D = \mathcal{O}_X/\mathcal{I}_D$)

$$0 \longrightarrow \mathcal{O}_X(-D) \otimes \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_X/(\mathcal{I}_C + \mathcal{I}_D) \longrightarrow 0$$

Thus the Euler characteristics satisfy

$$\chi(C, \mathcal{O}_C) - \chi(C, \mathcal{O}_X(-D)|_C) = \chi(X, \mathcal{O}_X/(\mathcal{I}_C + \mathcal{I}_D)) = h^0(X, \mathcal{O}_X/(\mathcal{I}_C + \mathcal{I}_D)) = C \cdot D,$$

since $\mathcal{O}_X/(\mathcal{I}_C + \mathcal{I}_D)$ is supported at finitely many points, so $h^1(X, \mathcal{O}_X/(\mathcal{I}_C + \mathcal{I}_D)) = 0$. Also,

$$\chi(C, \mathcal{O}_C) - \chi(C, \mathcal{O}_X(-D)|_C) = 0 - \deg(\mathcal{O}_X(-D)|_C) = \deg(\mathcal{O}_X(D)|_C)$$

by the Riemann-Roch theorem, which gives the desired formula. □

Example 25.0.4. If C is a smooth curve on X, then $\omega_C = \omega_X(C)|_C$, where $\omega_X(C) = \mathcal{O}_X(K_X + C)$. This implies that

$$2g(C) - 2 = \deg \omega_C = \deg(\omega_X(C)|_C) = (K_X + C) \cdot C.$$

As a special case, if C is a smooth degree d plane curve in $\mathbb{P}^2$, then we see that

$$2g(C) - 2 = (K_{\mathbb{P}^2} + C) \cdot C = (-3H + dH) \cdot dH = (d-3)d,$$

where $H \subseteq \mathbb{P}^2$ is a line. This recovers the formula $g(C) = (d-1)(d-2)/2$.

25.2 Riemann-Roch for Surfaces

Theorem 25.3 (Riemann-Roch for surfaces)**.** *For a divisor D on a smooth projective surface X,*

$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{2} D \cdot (D - K_X) + 1 + p_a(X),$$

where $p_a(X) = \chi(X, \mathcal{O}_X) - 1$ is the arithmetic genus *of X.*

Lemma 25.2. *Any divisor D on a smooth projective surface X can be written as $D \sim C' - C$ with C and C' smooth irreducible curves.*

Proof. Choose a very ample divisor H on X. For $k \gg 0$, the divisor $D + kH$ is very ample. By Bertini's theorem (see Mustata 6.4.1), there exist smooth irreducible curves $C \sim kH$ and $C' \sim D + kH$. Then we can write $D = (D + kH) - kH \sim C' - C$. □

Proof of Theorem 25.3. The goal is to reduce to the case of curves. Since the left and right hand sides only depend on the linear equivalence class of D, we may assume that $D = C - E$ with C, E smooth irreducible curves (by Lemma 25.2). We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(C - E) \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_E \longrightarrow 0$$

We also have a second short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_C \longrightarrow 0$$

Thus taking Euler characteristics, we see that

$$\begin{aligned}\chi(X, \mathcal{O}_X(C-E)) &= \chi(X, \mathcal{O}_X(C)) - \chi(X, \mathcal{O}_X(C) \otimes \mathcal{O}_E) \\ &= \chi(X, \mathcal{O}_X) + \chi(X, \mathcal{O}_X(C) \otimes \mathcal{O}_C) - \chi(X, \mathcal{O}_X(C) \otimes \mathcal{O}_E) \\ &= \chi(X, \mathcal{O}_X) + \chi(C, \mathcal{O}_X(C)|_C) - \chi(E, \mathcal{O}_X(C)|_E) \\ &= (1 + p_a(X)) + (C^2 + 1 - g(C)) - (C \cdot E + 1 - g(E)).\end{aligned}$$

Now $g(C) = \frac{1}{2} C \cdot (C + K_X) + 1$ and $g(E) = \frac{1}{2} E \cdot (E + K_X) + 1$, so we get

$$\chi(X, \mathcal{O}_X(C-E)) = \frac{1}{2}(C-E) \cdot (C - E - K_X) + 1 + p_a(X).$$

Since $D = C - E$, this gives the desired formula. $\square$

Remark. There is a criterion for ampleness in terms of intersection numbers (the *Nakai-Moishezon criterion*): A divisor D on X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves $C \subseteq X$. This is generally true in any dimension.

Lecture 26

Apr. 23 — Complex Analytic Geometry

26.1 Holomorphic Functions

Remark. Let $k = \mathbb{C}$. We will replace polynomials with holomorphic functions and consider the usual analytic topology on $\mathbb{C}^n$.

Definition 26.1. Let $U \subseteq \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is *holomorphic at* $c \in U$ (we say that f is *holomorphic* if it is holomorphic at every point) if one of the following equivalent conditions hold:

1. f is *analytic* at c;
2. f is *complex differentiable* at c;
3. the *Cauchy-Riemann equations* hold at c.

Definition 26.2. Let $U \subseteq \mathbb{C}^n$ be an open set. We say that a function $f : U \to \mathbb{C}$ is *holomorphic* if one of the following equivalent conditions hold:

1. f is analytic, i.e. if at each $c = (c_1, \dots, c_n) \in U$, there exist $a_J \in \mathbb{C}$ such that
$$f(z) = \sum_{J=(j_1,\dots,j_n)\in\mathbb{N}^n} a_J (z_1 - c_1)^{j_1} \cdots (z_n - c_n)^{j_n}$$
in some neighborhood around c;
2. at each $c = (c_1, \dots, c_n) \in U$, $f(c_1, \dots, c_{i-1}, z_i, c_{i+1}, \dots, c_n)$ is holomorphic in z_i near c_i,
3. the Cauchy-Riemann equations hold: Write $f(z) = f(x, y) = u(x, y) + iv(x, y)$, then
$$\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial y_i} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y_i} + \frac{\partial v}{\partial x_i} = 0.$$

Definition 26.3. A *germ* of a holomorphic function at $c \in \mathbb{C}^n$ is a holomorphic function $\varphi : U \to \mathbb{C}$ (where $c \in U \subseteq \mathbb{C}^n$) up to the equivalence relation where $(\varphi, U) \sim (\varphi', U')$ if there exists $c \in W \subseteq U \cap U'$ open such that $\varphi|_W = \varphi'|_W$. Define the space of germs
$$\mathcal{O}_n = \{\text{germs of holomorphic functions at } 0 \in \mathbb{C}^n\}.$$

Proposition 26.1. *We have the following:*

1. *$\mathcal{O}_n$ is a ring;*
2. *there is an inclusion $\mathcal{O}_n \to \mathbb{C}[\![z_1, \dots, z_n]\!]$ (we often write $\mathbb{C}\langle z_1, \dots, z_n\rangle$ for the image);*

3. *$\mathcal{O}_n$ has a unique maximal ideal $\mathfrak{m} = \{\varphi \in \mathcal{O}_n : \varphi(0) = 0\}$ (so $\mathcal{O}_n$ is a local ring);*
4. *$\mathcal{O}_n$ is Noetherian;*
5. *$\mathcal{O}_n$ is a UFD.*

Remark. When $n = 1$, the ideals of $\mathcal{O}_1$ are of the form (z^m) with $m \geq 0$ and (0). In general, proving Proposition 26.1(4)-(5) is more difficult and relies on the *Weierstrass division theorem.*

Remark. If we take an open set $U \subseteq \mathbb{C}^n$, we can consider

$$\mathcal{O}_{\mathbb{C}^n}(U) = \{\text{holomorphic functions } U \to \mathbb{C}\}.$$

However, $\mathcal{O}_{\mathbb{C}^n}(U)$ is not necessarily Noetherian. For example, in $\mathcal{O}_{\mathbb{C}}(\mathbb{C})$ we can take

$$I_k = \{\varphi \in \mathcal{O}_{\mathbb{C}}(\mathbb{C}) : f(i) = 0 \text{ for all } i \in \mathbb{N} \text{ with } i \geq k\}.$$

Then $I_k \subseteq I_{k+1} \subseteq I_{k+2} \subseteq \cdots$ and the chain is strict: Something like $\sin(\pi z)/(z-k)$ lies in $I_{k+1} \setminus I_k$.

26.2 Analytic Subvarieties

Definition 26.4. Let $U \subseteq \mathbb{C}^n$ be an open set. An *analytic subvariety* of U is a subset $X \subseteq U$ such that for each $x \in X$, there exists an open neighborhood $x \in V \subseteq U$ such that

$$V \cap X = \{x \in V : f_1(x) = \cdots = f_r(x) = 0\} =: V(f_1, \ldots, f_r)$$

for some $f_1, \ldots, f_r \in \mathcal{O}_{\mathbb{C}^n}(V)$.

Example 26.4.1. The following are analytic subvarieties:

1. Hypersurfaces: Given $f \in \mathcal{O}_{\mathbb{C}^n}(U)$, we have $V(f) = \{x \in U : f(x) = 0\}$.
2. $Z = V(w - e^z) \subseteq \mathbb{C}^2$. Note that $Z \cap V(w-1) = \{(1, 2\pi i k) : k \in \mathbb{Z}\}$, which is notably infinite.[1]
3. Let $U = \mathbb{C}^2_{w,z} \setminus \{w = 0\}$ and $X = \bigcup_{n=1}^{\infty} V(z - nw)$. Note that $\overline{X} \subseteq \mathbb{C}^2$ is *not* an analytic subvariety of $\mathbb{C}^2$. This is intuitively because $\operatorname{codim}_{\mathbb{C}^2}\{w = 0\} = 1$ and $\operatorname{codim}_U X = 1$.

Definition 26.5. A *germ* of a complex analytic subvariety of $\mathbb{C}^n$ at 0 is a complex analytic subvariety $0 \in X \subseteq U \subseteq \mathbb{C}^n$ (where U is open in $\mathbb{C}^n$), up to the equivalence relation where $X \sim X'$ if there exists $0 \in W \subseteq U \cap U'$ such that $X \cap W = X' \cap W$.

Remark. Given a germ $0 \in X \subseteq \mathbb{C}^n$ of a complex analytic subvariety, we can define

$$I(X) = \{\varphi \in \mathcal{O}_n : \varphi \text{ vanishes on a neighborhoods of } X \text{ containing } 0\}.$$

Similarly, given an ideal $I \subseteq \mathcal{O}_n$, we can define

$$0 \in X = V(f_1, \ldots, f_r),$$

where $f_1, \ldots, f_r$ are generators of I (recall that $\mathcal{O}_n$ is Noetherian) defined on some open neighborhood of $0 \in U \subseteq \mathbb{C}^n$. Note that X itself is not an analytic subvariety, but a germ of an analytic subvariety.

Proposition 26.2. *We have the following:*

1. *if $X_1 \subseteq X_2$ are germs of analytic subvarieties (i.e. there is some neighborhood $0 \in U \subseteq \mathbb{C}^n$ where $X_1 \cap U \subseteq X_2 \cap U$), then $I(X_1) \supseteq I(X_2)$;*

[1] We saw that with polynomials, the intersection of two distinct curves is finite.

2. *if $I_1 \subseteq I_2 \subseteq \mathcal{O}_n$ are ideals, then $V(I_1) \supseteq V(I_2)$;*
3. *$V(I(X)) = X$;*
4. *$I(V(I)) = \sqrt{I}$;*
5. *X is irreducible if and only if $I(X)$ is prime;*
6. *any germ of a complex analytic subvariety at $0 \in \mathbb{C}^n$ can be written uniquely (up to reordering) as a union of irreducible ones.*

Remark. Consider the curve $x^2 - y^2 - y^3 = 0$. Recall that this is irreducible in algebraic geometry (i.e. in the Zariski topology). However, in complex analytic geometry, we can write

$$x^2 - y^2 - y^3 = x^2 - y^2(1+y) = (x - y\sqrt{1+y})(x + y\sqrt{1+y})$$

for $|y| < 1$, so this curve is not irreducible as a germ of a complex analytic subvariety.

Remark. The finiteness of the irreducible decomposition fails for non-germs: Consider

$$\{0, 1, 2, \dots\} \subseteq \mathbb{C}.$$

26.3 Complex Manifolds

Definition 26.6. A *complex manifold* is a locally ringed space $(X, \mathcal{O}_X)$ such that

- X is Hausdorff with countable basis;
- $\mathcal{O}_X$ is a subsheaf of the sheaf of $\mathbb{C}$-valued continuous functions on X;
- at any $x \in X$, there exists $x \in U \subseteq X$ open such that

  $$(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_{\mathbb{C}^n}|_V)$$

  for some open set $0 \in V \subseteq \mathbb{C}^n$.

Example 26.6.1. The following are examples of complex manifolds:

1. $X = \mathbb{C}^n$ or $\mathbb{CP}^n$.
2. $X = V(f) \subseteq \mathbb{C}^n$ such that $f \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ and $[\partial f/\partial z_1, \cdots, \partial f/\partial z_n] \neq 0$ on X.

Definition 26.7. A *complex analytic closed subvariety* of a complex manifold X is a closed set $Z \subseteq X$ that is locally a complex analytic subvariety for an open ball $V \subseteq \mathbb{C}^n$.

Theorem 26.1 (Chow's theorem)**.** *If $X \subseteq \mathbb{CP}^n$ is an analytic subvariety, then there exist homogeneous polynomials $F_1, \dots, F_r \in \mathbb{C}[z_0, \dots, z_n]$ such that $X = V(F_1, \dots, F_r)$.*

Proof. Consider the quotient map $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Note that $Z = q^{-1}(X)$ is a complex analytic subvariety of $\mathbb{C}^{n+1} \setminus \{0\}$. Now we have

$$\operatorname{codim}_{\mathbb{C}^{n+1}} \{0\} = n + 1 > \operatorname{codim}_{\mathbb{C}^{n+1} \setminus \{0\}} Z,$$

so Levi's extension theorem says that $\overline{Z}$ is a complex analytic subvariety of $\mathbb{C}^{n+1}$. Let $I = I(\overline{Z}) \subseteq \mathcal{O}_{n+1}$. Now for any $f \in I$, we can write $f(z) = \sum_{j \geq 0} f_j(z)$ with f_j a homogeneous polynomial of degree j.

For any fixed $a \in \overline{Z}$, we have

$$0 = f(\lambda a) = \sum_{j \geq 0} \lambda^j f_j(a)$$

for every $\lambda \in \mathbb{C}$, so $f_j(a) = 0$ for all $j \geq 0$. So $f_j \in I$. Thus there are homogeneous polynomials $(F_j)_{j \in J}$ that generate I (up to infinite sums). This gives that $V(F_j : j \in J) = X$, and we can then choose a finite collection cutting out X by Noetherianity. □

Example 26.7.1. Consider $X = \{1, 2, \ldots, \} \subseteq \mathbb{C}^1 \subseteq \mathbb{CP}^1$. Now X is an analytic subvariety of $\mathbb{C}^1$, but it is not an analytic subvariety of $\mathbb{CP}^1$.

Lecture 27

Apr. 28 — GAGA

27.1 Revisiting Coherent Sheaves

Remark. Recall Chow's theorem says that if $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a closed analytic subvariety, then there exist $f_0, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous such that $X = V(f_0, \ldots, f_r)$. Our goal is to compare coherent sheaves on X^{alg} and X^{an}. This problem was solved by Serre in his GAGA paper.

Remark. Let $(X, \mathcal{O}_X)$ be a ringed space, e.g. an algebraic variety or a complex analytic variety.

Definition 27.1. Let $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$.

1. $\mathcal{F}$ is *finite type* if at each $x \in X$, there exists $x \in U \subseteq X$ open and a surjection $\mathcal{O}_U^{\oplus r} \twoheadrightarrow \mathcal{F}|_U$ for some $r \geq 0$.
2. $\mathcal{F}$ is *coherent* if it is finite type and for every $U \subseteq X$ and morphism $\phi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$, we have that $\ker \phi$ is finite type (on U).

Remark. If $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ is coherent, then at each $x \in X$, there exists $x \in U \subseteq X$ open such that there exists an exact sequence

$$\mathcal{O}_U^{\oplus s} \longrightarrow \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

Example 27.1.1. We have the following:

1. $\mathcal{O}_X^{\oplus r}$ is finite type.
2. If $(X, \mathcal{O}_X)$ is an algebraic variety, then this new definition of coherence agrees with the old one.

Theorem 27.1 (Oka-Cartan)**.** *If $(X, \mathcal{O}_X)$ is a complex analytic space, then $\mathcal{O}_X$ is coherent.*

Proof. Look this up online, the proof uses the Weierstrass division theorem. Oka showed that $\mathcal{O}_{\mathbb{C}^n}$ is coherent, and Cartan showed that if $X \subseteq \mathbb{C}^n$ is a complex analytic subvariety, then $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{C}^n}$ is coherent (recall that $i_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{C}^n}/\mathcal{I}_X$ where $i : X \hookrightarrow \mathbb{C}^n$). □

Remark. The category Coh_X of coherent sheaves on X is an abelian category.

Definition 27.2. Let $\mathcal{F} \in \mathrm{Coh}_X$. Define

$$H^i(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^i(\mathcal{U}, \mathcal{F}),$$

where $\mathcal{U}$ ranges through all open (not necessarily affine) covers of X.

Remark. We have the following:

1. A short exact sequence in Coh_X gives rise to a longe exact sequences on H^i.
2. If $(X, \mathcal{O}_X)$ is a separated algebraic variety, then this new definition of H^i agrees with the old one.

27.2 GAGA

Remark. Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a projective algebraic variety, and let $X^{\mathrm{an}} \subseteq \mathbb{P}^{n,\mathrm{an}}_{\mathbb{C}}$ be X viewed as a projective complex analytic variety. Note that there is a morphism of ringed spaces

$$(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \longrightarrow (X, \mathcal{O}_X)$$

where $X^{\mathrm{an}} \to X$ is a set-theoretic bijection and $\mathcal{O}_X \to i_*\mathcal{O}_{X^{\mathrm{an}}}$ is given on open sets by $f \mapsto f^{\mathrm{an}}$. Letting i^{-1} be the adjoint functor to i_*, we get a functor

$$\begin{aligned} \Phi : \mathrm{Coh}_X &\longrightarrow \mathrm{Coh}_{X^{\mathrm{an}}} \\ \mathcal{F} &\longmapsto i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}, \end{aligned}$$

where the right-hand side is coherent by the Oka-Cartan theorem.

Remark. The functor Φ is exact. To prove this, Serre uses that the map

$$\mathcal{O}_{X,x} \longrightarrow \widehat{\mathcal{O}}_{X,x}$$

(here $\widehat{\mathcal{O}}_{X,x}$ is the completion of $\mathcal{O}_{X,x}$) is faithfully flat and $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{X^{\mathrm{an}},x}$.

Remark. If $\mathcal{U}$ is an open cover of X, then we get a map

$$H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{U}^{\mathrm{an}}, \Phi\mathcal{F}), .$$

This induces a natural map (after taking direct limits)

$$\phi^i_{\mathcal{F}} : H^i(X, \mathcal{F}) \longrightarrow H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}).$$

Theorem 27.2 (Theorem A). *If $\mathcal{F} \in \mathrm{Coh}_X$, then $\phi^i_{\mathcal{F}}$ is an isomorphism.*

Proof. The sketch of the proof is as follows:

0. Reduce to the case $X = \mathbb{P}^n$.
1. Prove the result for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}$.

   For this part, one notes that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \mathbb{C} = H^0(\mathbb{P}^{n,\mathrm{an}}, \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}})$ and for $i \geq 1$,

   $$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 = H^i(\mathbb{P}^{n,\mathrm{an}}, \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}}).$$

   The equality $H^i(\mathbb{P}^{n,\mathrm{an}}, \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}}) = 0$ for $i \geq 1$ follows from *Hodge theory.*
2. Prove the result for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(m)$.

   To prove this, use the short exact sequence (let $H = V(x_0)$)

   $$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(m-1) \xrightarrow{\times x_0} \mathcal{O}_{\mathbb{P}^n}(m) \longrightarrow \mathcal{O}_H(m) \longrightarrow 0$$

   Then use induction on n to see that $\phi^i_{\mathcal{O}(m)}$ is an isomorphism if and only if $\phi^i_{\mathcal{O}(m-1)}$ is.

3. Prove the result for $\mathcal{F}$ arbitrary.

   To do this, one chooses a presentation

$$\bigoplus_{j=1}^{s} \mathcal{O}_{\mathbb{P}^n}(n_j) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(m_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

   and uses the previous steps.

See Serre's GAGA paper for more details. □

Theorem 27.3 (Theorem B)**.** *For $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}_X$, the map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$$

is an isomorphism.

Proof. Set $\mathcal{A} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $\mathcal{B} := \mathcal{H}om_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$. Then the result is equivalent to showing that $\Gamma(X, \mathcal{A}) \to \Gamma(X^{\mathrm{an}}, \mathcal{B})$ is an isomorphism. Consider the analytification of $\mathcal{A}$:

$$\begin{tikzcd}
\Gamma(X, \mathcal{A}) \arrow[r] \arrow[dr, "\cong"'] & \Gamma(X^{\mathrm{an}}, \mathcal{B}) \\
 & \Gamma(X^{\mathrm{an}}, \mathcal{A}^{\mathrm{an}}) \arrow[u]
\end{tikzcd}$$

The labeled map is an isomorphism by Theorem 27.2, so it suffices to show $\mathcal{A}^{\mathrm{an}} = \mathcal{B}$. See GAGA. □

Remark. Both Theorem 27.2 and Theorem 27.3 may be false if X is not assumed to be projective. For example, consider $X = \mathbb{A}^1$ and $\mathcal{F} = \mathcal{G} = \mathcal{O}_X$.

Theorem 27.4 (Theorem C)**.** *For any $\mathcal{H} \in \mathrm{Coh}_{X^{\mathrm{an}}}$, there exists $\mathcal{F}$ such that $\mathcal{F}^{\mathrm{an}} \cong \mathcal{H}$. In particular,*

$$\Phi : \mathrm{Coh}_X \longrightarrow \mathrm{Coh}_{X^{\mathrm{an}}}$$

is an equivalence of categories.

Proof. The idea is to show $\mathcal{H}(m)$ is globally generated for $m \gg 0$. Use this to construct a presentation

$$\bigoplus_{j=1}^{s} \mathcal{O}_{X^{\mathrm{an}}}(n_j) \xrightarrow{\alpha} \bigoplus_{i=1}^{r} \mathcal{O}_{X^{\mathrm{an}}}(m_i) \longrightarrow \mathcal{H} \longrightarrow 0$$

and use Theorem 27.3 to show that α is algebraic. □

Remark. Theorem 27.4 gives an alternative proof of Chow's theorem (Theorem 26.1). Fix an analytic subvariety $Z \subseteq \mathbb{P}^n$. Then Cartan's theorem implies $\mathcal{I}_Z$ is coherent, and Theorem 27.4 implies there exists $\mathcal{F} \in \mathrm{Coh}_{\mathbb{P}^n}$ such that $\mathcal{F}^{\mathrm{an}} \cong \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}}/\mathcal{I}_Z$. Now $\mathrm{Supp}\, \mathcal{F} = \mathrm{Supp}\, \mathcal{F}^{\mathrm{an}} = Z$.

Lecture 28

Apr. 30 — Hodge Theory

28.1 Hodge Decomposition Theorem

Remark. For this lecture, we will always denote Cech cohomology by $\check{H}^i$. We want to see how this is related to the singular cohomology $H^i_{\text{sing}}(X^{\text{an}}, \mathbb{C})$.

Theorem 28.1 (Hodge decomposition theorem)**.** *For $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ a smooth projective variety of dimension n, there exists a decomposition*

$$H^k_{\text{sing}}(X^{\text{an}}, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

for $0 \leq k \leq 2n$. Furthermore,

1. *$\overline{H^{p,q}} = H^{q,p}$ (where we view $H^k_{\text{sing}}(X^{\text{an}}, \mathbb{C}) = H^k_{\text{sing}}(X^{\text{an}}, \mathbb{R})$ by the universal coefficients theorem and $\overline{H^{p,q}}$ denotes complex conjugation);*
2. *$H^{p,q} = H^q(X, \Omega^p_X)$ ($= H^q(X^{\text{an}}, \Omega^{p,\text{an}}_X)$ by GAGA);*
3. *writing $h^{p,q} = \dim H^{p,q}$, we have $h^{p,q} = h^{q,p}$ (by (1), also called* Hodge duality*) and $h^{p,q} = h^{n-p,n-q}$ (by Serre duality).*

Remark. Note that $b_i = \dim H^i_{\text{sing}}(X^{\text{an}}, \mathbb{C}) = \sum_{p+q=i} h^{p,q}$, so the Betti numbers b_i are even for i odd.

Example 28.0.1. Let C be a genus g curve. Then for $k = 1$, we have

$$h^{1,1} = h^{0,0} = h^0(C, \mathcal{O}_C) = 1.$$

Note that we have the diamond

$$\begin{array}{ccc} & h^{1,1} = 1 & \\ h^{0,1} = g & & h^{1,0} = g \\ & h^{0,0} = 1 & \end{array}$$

Remark. We want to explain where the $H^{p,q}$ come from and why it is related to $H^q(X, \Omega^p_X)$.

28.2 Cohomology of Sheaves

Remark. Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups (e.g. X a complex manifold with $\mathcal{F} = \underline{\mathbb{C}}$ or $\mathcal{O}_X$). Recall that we have defined Cech cohomology, which we now denote by $\check{H}^i(X, \mathcal{F})$.

Definition 28.1. A sheaf $\mathcal{F}$ is *flasque* if $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective for all $V \subseteq U \subseteq X$ open.

Example 28.1.1. Let $\mathcal{G}$ be a sheaf. Define the *Godement sheaf* $\mathrm{God}(\mathcal{G})$ by

$$U \longmapsto \mathrm{God}(\mathcal{G})(U) := \prod_{x \in U} \mathcal{G}_x.$$

Then one can observe that $\mathrm{God}(\mathcal{G})$ is flasque.

Definition 28.2. A *flasque resolution* of a sheaf $\mathcal{F}$ is a complex of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \cdots$$

such that

1. $d^2 = 0$;
2. $\operatorname{im} d_{i-1} = \ker d_i$ for all i;
3. each $\mathcal{I}^i$ is flasque.

Example 28.2.1 (Godement resolution)**.** Let $\mathcal{F}$ be a sheaf. Then

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "\alpha"] & \mathrm{God}(\mathcal{F}) \arrow[r, "d_0"] \arrow[d, dashed] & \mathrm{God}(\operatorname{coker}\alpha) \arrow[r] \arrow[d, dashed] & \cdots \\
& & \operatorname{coker}\alpha \arrow[ur, dashed] & \operatorname{coker} d_0 \arrow[ur, dashed] &
\end{tikzcd}$$

is a (canonical) flasque resolution of $\mathcal{F}$.

Definition 28.3 (Sheaf cohomology)**.** Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups. Then we can define

$$H^i(X, \mathcal{F}) := H^i(\Gamma(X, \mathcal{I}^\bullet)),$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is a flasque resolution of $\mathcal{F}$ (e.g. the Godement resolution). One can check that this definition is independent of the choice of flasque resolution.

Example 28.3.1. If X is locally contractible (e.g. a manifold), then

$$H^k(X, \underline{\mathbb{R}}) \cong H^k_{\mathrm{sing}}(X, \mathbb{R}).$$

To see this, consider the complex of sheaves

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{C}^0_{\mathrm{sing}} \xrightarrow{\partial} \mathcal{C}^1_{\mathrm{sing}} \xrightarrow{\partial} \cdots$$

where $\mathcal{C}^p_{\mathrm{sing}}$ is the sheaf defined by $U \mapsto C^p_{\mathrm{sing}}(U, \mathbb{R})$. We can note that:

1. $\partial^2 = 0$ (from algebraic topology);
2. the sequence is exact (since X is locally contractible);
3. each $\mathcal{C}^p_{\mathrm{sing}}$ is flasque (essentially by the definition of a cochain).

Thus this complex is a flasque resolution, so we see that

$$H^k(X, \underline{\mathbb{R}}) = \frac{\ker(\partial : C^k_{\mathrm{sing}}(X, \mathbb{R}) \to C^{k+1}_{\mathrm{sing}}(X, \mathbb{R}))}{\operatorname{im}(\partial : C^{k-1}_{\mathrm{sing}}(X, \mathbb{R}) \to C^k_{\mathrm{sing}}(X, \mathbb{R}))} = H^k_{\mathrm{sing}}(X, \mathbb{R}).$$

In fact, this works more generally for any ring R, not just $\mathbb{R}$.

Theorem 28.2. *If X is Hausdorff and paracompact, then*

$$\check{H}^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})$$

is an isomorphism.

Example 28.3.2. Let X be a smooth manifold. Consider the de Rham complex

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{A}^1 \longrightarrow \cdots$$

where $\mathcal{A}^p$ is the sheaf defined by $U \mapsto \mathcal{A}^p(U)$ (the space of smooth p-forms on U). We have:

1. exactness: this is by the Poincaré lemma;
2. $d^2 = 0$: this follows from differentiation rules;
3. flasqueness: this fails, even for $\mathcal{A}^0$.

However, $\mathcal{A}^p$ is still a *fine* sheaf, so by Theorem 28.3 below, we still have

$$H^p(X, \underline{\mathbb{R}}) = \frac{\ker(d : \mathcal{A}^p(X) \to \mathcal{A}^{p+1}(X))}{\operatorname{im}(d : \mathcal{A}^{p-1}(X) \to \mathcal{A}^p(X))} = H^p_{\mathrm{dR}}(X, \mathbb{R}).$$

Definition 28.4. A sheaf $\mathcal{F}$ is *fine* if it can be viewed as a sheaf of $\mathcal{A}$-modules for some sheaf of rings $\mathcal{A}$ on X such that for every open cover $\{U_i\}_{i\in I}$ of X, there exist $f_i \in \mathcal{A}(U_i)$ such that $\sum_{i\in I}(f_i)_x = 1$.

Example 28.4.1. Each $\mathcal{A}^p$ is fine, as we can view $\mathcal{A}^p$ as a sheaf of $\mathcal{A}$-modules, where $\mathcal{A}$ is the sheaf of smooth functions. Note that $\mathcal{A}$ admits partitions of unity.

Theorem 28.3. *If $\mathcal{F}$ is a sheaf and admits a resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ by fine sheaves, then*

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet)).$$

28.3 Complex Geometry

Remark. Let X be a complex manifold of complex dimension n and $\mathcal{A}^k_{\mathbb{C}}$ the sheaf of complex k-forms. Locally, with respect to complex coordinates $z_1, \ldots, z_n$ with $z_j = x_j + iy_j$, a complex k-form is given by

$$\sum_{\substack{I,J\subseteq\{1,\ldots,n\}\\ |I|+|J|=k}} f_{I,J}\, dx_{i_1} \wedge \cdots \wedge dx_{i_r} \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_s},$$

where the $f_{I,J}$ are smooth complex-valued functions. Then we have a complex

$$0 \longrightarrow \underline{\mathbb{C}} \xrightarrow{d} \mathcal{A}^0_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^1_{\mathbb{C}} \xrightarrow{d} \cdots$$

which is a fine resolution of $\underline{\mathbb{C}}$. Thus we have $H^k(X, \underline{\mathbb{C}}) \cong H^k_{\mathrm{dR}}(X, \mathbb{C})$.

Remark. Observe that there is a decomposition

$$\mathcal{A}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q},$$

where $\mathcal{A}^{p,q}$ is the sheaf of (p,q)-forms. Writing $z_j = x_j + iy_j$ and $\overline{z}_j = x_j - iy_j$, we can locally write a (p,q)-form in the form

$$\sum_{|I|=p,|J|=q} f_{I,J}\, dz_I \wedge d\overline{z}_J,$$

with $f_{I,J}$ smooth complex-valued functions. Then $d : \mathcal{A}^k_{\mathbb{C}} \to \mathcal{A}^{k+1}_{\mathbb{C}}$ decomposes as $d = \partial + \overline{\partial}$, where

$$\partial(f_{I,J}\, dz_I \wedge d\overline{z}_J) = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} f_{I,J}\, dz_i \wedge dz_I \wedge d\overline{z}_J$$

and $\overline{\partial}$ is defined similarly. This decomposition satisfies the following properties:

1. $\partial^2 = \overline{\partial}^2 = 0$;
2. $\partial\overline{\partial} + \overline{\partial}\partial = 0$;
3. $\partial\mathcal{A}^{p,q} \subseteq \mathcal{A}^{p+1,q}$ and $\overline{\partial}\mathcal{A}^{p,q} \subseteq \mathcal{A}^{p,q+1}$;
4. $\mathcal{O}_X = \ker(\overline{\partial} : \mathcal{A}^{0,0} \to \mathcal{A}^{0,1})$ and $\Omega^p_X = \ker(\overline{\partial} : \mathcal{A}^{p,0} \to \mathcal{A}^{p,1})$.[1]

Now the complex

$$0 \longrightarrow \Omega^p_X \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1} \xrightarrow{\overline{\partial}} \cdots$$

satisfies the following:

1. $A^{p,q}$ is fine ($\mathcal{O}_X$ has partitions of unity);
2. exactness (this follows by the antiholomorphic Poincaré lemma).

Thus we get that

$$H^q(X, \Omega^p_X) = H^q(\Gamma(X, \mathcal{A}^{p,\bullet})) = \frac{\ker(\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\operatorname{im}(\overline{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))} =: H^{p,q}.$$

We call the $H^{p,q}$ the *Dolbeault cohomology* of X. Then the Hodge decomposition theorem says that

$$H^k_{\text{sing}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

(in the case that $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a smooth projective variety).

[1] Here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions and Ω^p_X denotes the sheaf of holomorphic p-forms. Note that Ω^p_X coincides with the analytification of the sheaf of algebraic p-forms when $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a projective algebraic variety.